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A sharp estimate of the k-modulus of smoothness of Bessel potentials: an application to optimal embeddings

Let $X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be a rearrangement-invariant Banach function space over the measure space $(\mathbb{R}^n, \mu_n)$, where μ_n stands for the n-dimensional Lebesgue measure in $\mathbb{R}^n$. We derive a sharp estimate for the k-modulus of smoothness of the convolution of a function $f \in X(\mathbb{R}^n)$ with the Bessel potential kernel g_σ, $\sigma \in (0, n)$. Such an estimate states that if g_σ belongs to the associate space of X, then

$$\omega_k(f * g_\sigma, t) \precsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \text{for all} \quad t \in (0,1) \quad \text{and every} \quad f \in X(\mathbb{R}^n)$$

provided that $k \geq [\sigma]+1$ (f^* denotes the non-increasing rearrangement of f). One of the key steps in the proof of the sharpness of this estimate is the assertion that $\operatorname{sgn} \frac{\partial^j g_\sigma}{\partial x_1^j}(x) = (-1)^j$, $\sigma \in (0,n)$, $j \in \mathbb{N}$, for all x in a small circular half-cone whose vertex is at the origin and whose axis coincides with the positive part of x_1-axis.

The above estimate enables us to derive optimal continuous embeddings of Bessel potential spaces $H^\sigma X(\mathbb{R}^n)$, where, in limiting situations, we are able to obtain embeddings into Zygmund-type spaces rather than Hölder-type spaces. In particular, such results show that the Brézis-Wainger embedding of the Sobolev space $W^{k+1,n/k}(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $k < n-1$, into the space of "almost" Lipschitz functions, is a consequence of a better embedding whose target is a Zygmund-type space.

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