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Sharp conditions on multilinear Fourier multipliers

This talk is based on joint works with Naohito Tomita (Osaka University) and with Loukas Grafakos (University of Missouri). For $m \in L^\infty(\mathbb{R}^{2n})$, the bilinear Fourier multiplier operator T_m is defined by

$$T_m(f_1, f_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix(\xi_1+\xi_2)} m(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2)\, d\xi_1 d\xi_2,$$

where $f_1,\ f_2 \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Coifman-Meyer (1, 2, 3) proved that if the multiplier $m(\xi_1, \xi_2)$ satisfies the condition

$$|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)| \le C_{\alpha_1,\alpha_2} (|\xi_1| + |\xi_2|)^{-|\alpha_1|-|\alpha_2|}, \tag{1}$$

then T_m extends to a bounded operator $L^{p_1} \times L^{p_2} \to L^p$ for $p_1,\ p_2,\ p$ satisfying $1 < p_1,\ p_2,\ p \le \infty$ and $1/p_1 + 1/p_2 = 1/p$, where $L^{p_1} \times L^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to BMO$ if $p_1 = p_2 = p = \infty$. The results were extended to the full range $0 < p_1,\ p_2,\ p \le \infty$ by the works of Kenig-Stein (5) and Grafakos-Kalton (4). To assure the boundedness of T_m, it is sufficient to assume (1) for derivatives up to certain order. In this talk we shall consider the problem to find the differentiability conditions of the type (1) that are "as small as possible" to assure the boundedness of T_m.
To measure the smoothness of multipliers, we use the product type Sobolev norm, which is defined as follows. For $s_1,\ s_2 > 0$ and for $F \in \mathcal{S}'(\mathbb{R}^{2n})$, the norm $\|F\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}$ is defined by

$$\|F\|_{W^{(s_1,s_2)}} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|\xi_1|)^{2s_1} (1+|\xi_2|)^{2s_2} |\widehat{F}(\xi_1, \xi_2)|^2\, d\xi_1 d\xi_2 \right)^{1/2}.$$

We take a function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ that satisfies $\operatorname{supp} \Psi \subset \{\xi \in \mathbb{R}^{2n} : 1/2 \le |\xi| \le 2\}, \quad \sum_{j\in\mathbb{Z}} \Psi(\xi/2^j) = 1$ for all $\xi \in \mathbb{R}^{2n} \setminus \{0\}$, and we define

$$A_{s_1,s_2}(m) = \sup_{j\in\mathbb{Z}} \|m(2^j\xi_1, 2^j\xi_2)\Psi(\xi_1, \xi_2)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}.$$

The condition $A_{s_1,s_2}(m) < \infty$ could be understood as m satisfies the condition (1) in the sense of L^2-derivatives up to order s_i with respect to ξ_i, $i = 1, 2$. We write $\|T_m\|_{H^{p_1}(\mathbb{R}^n)\times H^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ to denote the smallest constant C that satisfies $\|T_m(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C\|f_1\|_{H^{p_1}(\mathbb{R}^n)}\|f_2\|_{H^{p_2}(\mathbb{R}^n)}$ for all $f_i \in \mathcal{S}(\mathbb{R}^n) \cap H^{p_i}(\mathbb{R}^n)$, where H^{p_i} are the Hardy spaces and by convention $H^{p_i} = L^{p_i}$ if $1 < p_i \le \infty$. We define $\|T_m\|_{L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)\to BMO(\mathbb{R}^n)}$ in the same way. The main results are the following.

Theorem 1 *Let $0 < p_1,\ p_2,\ p \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. If*

$$s_1 > \max\{n/2, n/p_1 - n/2\}, \quad s_2 > \max\{n/2, n/p_2 - n/2\}, \quad s_1 + s_2 > n/p_1 + n/p_2 - n/2,$$

then

$$\|T_m\|_{H^{p_1}(\mathbb{R}^n)\times H^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim A_{s_1,s_2}(m), \tag{2}$$

where $H^{p_1} \times H^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to BMO$ if $p_1 = p_2 = p = \infty$.

Theorem 2 *Let $0 < p_1,\ p_2,\ p \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. Then the estimate (2), where $H^{p_1} \times H^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to BMO$ if $p_1 = p_2 = p = \infty$, holds only if*

$$s_1 \ge \max\{n/2, n/p_1 - n/2\}, \quad s_2 \ge \max\{n/2, n/p_2 - n/2\}, \quad s_1 + s_2 \ge n/p_1 + n/p_2 - n/2.$$

Some results on multilinear Fourier multipliers will also be mentioned.

References

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