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Elliptic operators and function spaces on quasi-bounded domains

An unbounded domain Ω in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega, |x|\to\infty} \operatorname{dist}(x, \partial\Omega) \, = \, 0 \ .$$

We consider the Besov and Triebel-Lizorkin spaces defined on a wide range of quasi-bounded domains. We obtain sufficient and necessary conditions for compactness of the Sobolev embeddings. Moreover we study the degree of the compactness of the embeddings in terms of entropy numbers. In particular for the quasi-bounded domains $\Omega \subset \mathbb{R}^n$ we define the box packing constant $b(\Omega)$, $n \le b(\Omega) \le \infty$, and prove that the embedding $\widetilde{W}^{k_1}_{p_1}(\Omega) \hookrightarrow \widetilde{W}^{k_2}_{p_2}(\Omega)$ is compact if $k_1 - k_2 - n\big(\frac{1}{p_1} - \frac{1}{p_2}\big) > b(\Omega)\big(\frac{1}{p_2} - \frac{1}{p_1}\big)_+$. Here $\widetilde{W}^{k}_{p}(\Omega)$ is a function space obtained by completing $C_0^\infty(\Omega)$ in the usual Sobolev norm. Moreover we prove that if the embedding is compact and $b(\Omega) < \infty$, then

$$e_k\Big(\widetilde{W}^{k_1}_{p_1}(\Omega) \hookrightarrow \widetilde{W}^{k_2}_{p_2}(\Omega)\Big) \sim k^{-\gamma} \qquad \text{with} \qquad \gamma = \frac{k_1 - k_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) .$$

The inverse entropy problem is also considered. We apply the above estimates to spectral properties of elliptic operators. Let

$$A(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) \partial^\alpha$$

be a self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \le 2m$. Let $\lambda_1, \lambda_2, \ldots$ be eigenvalues of A ordered by their magnitude and counted according to their multiplicities. Then

$$\lambda_k \sim k^{\frac{2m}{b(\Omega)}} \ , \quad k \in \mathbb{N} \ .$$

This is joint work with H.-G.Leopold (Jena).