Sobolev Spaces on Metric Measure Spaces

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(Joint work)

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Outline

I. Characterizations of Hajłasz-Sobolev spaces via the **grand Littlewood-Paley functions**

II. **Pointwise** characterizations of Besov and Triebel-Lizorkin spaces

III. **Pointwise** characterizations of **Newton**-Besov and **Newton**-Triebel-Lizorkin spaces

IV. Further remarks

I. Characterizations of Hajłasz-Sobolev spaces via the grand Littlewood-Paley functions

Function Spaces / I

$$\textbf{Function Spaces} \begin{cases} \mathrm{BMO}(\mathbb{R}^n)\ (L^\infty(\mathbb{R}^n)),\ p=\infty; \\ \\ L^p(\mathbb{R}^n),\quad p\in(1,\infty); \\ \\ H^p(\mathbb{R}^n)\quad (L^p(\mathbb{R}^n)),\quad p\in(0,1]. \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{array}{c}\textbf{Sobolev spaces} \\ \dot{W}^{m,p}(\mathbb{R}^n)\end{array} \Longrightarrow \begin{cases} \textbf{Besov spaces } \dot{B}^s_{p,q}(\mathbb{R}^n), \\ \textbf{Triebel – Lizorkin spaces} \\ \qquad \dot{F}^s_{p,q}(\mathbb{R}^n). \end{cases}$$

- $\mathrm{BMO}(\mathbb{R}^n)=\dot{F}^0_{\infty,2}(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)=\dot{F}^0_{p,2}(\mathbb{R}^n),\ p\in(0,\infty)$ & $\dot{W}^{m,p}(\mathbb{R}^n)=\dot{F}^m_{p,2}(\mathbb{R}^n),\ m\in\mathbb{Z}_+\equiv\{0\}\cup\mathbb{N},\ p\in(1,\infty)$.

Lebesgue Spaces / I

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space.

- **Lebesgue spaces** $L^p(\mathbb{R}^n)$, $p \in (0, \infty]$

$f \in L^p(\mathbb{R}^n) \Longleftrightarrow f$ is Lebesgue measurable, and

$$\|f\|_{L^p(\mathbb{R}^n)} \equiv \left\{\int_{\mathbb{R}^n} |f(x)|^p\,dx\right\}^{1/p} < \infty, \quad p \in (0, \infty),$$

and

$$\|f\|_{L^\infty(\mathbb{R}^n)} \equiv \operatorname*{esssup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

(Only characterize the **size** of functions)

Sobolev Spaces (1) / I

- **Sobolev spaces** $\dot{W}^{m,p}(\mathbb{R}^n)$ & $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, $p \in (1, \infty)$

  $f \in \dot{W}^{m,p}(\mathbb{R}^n) \Longleftrightarrow f \in \mathcal{S}'(\mathbb{R}^n)$ (**Schwartz distribution**) and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| = m$; moreover

$$\|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)} \equiv \sum_{|\gamma|=m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

Sobolev Spaces (2) / I

$f \in W^{m,p}(\mathbb{R}^n) \Longleftrightarrow f \in L^p(\mathbb{R}^n)$ and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| \le m$; moreover

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} \equiv \sum_{|\gamma|\le m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

$$\left(\|f\|_{W^{m,p}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)}\right)$$

(Characterize the **smooth** properties of functions)

Sobolev Spaces — A Nice Observation (1) / I

- For any $R \in (0, \infty]$, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let (**the local Hardy-Littlewood maximal function**)

$$M_R(g)(x) \equiv \sup_{r<R} \frac{1}{r^n} \int_{B(x,r)} |g(y)|\, dy,$$

where $B(x,r) \equiv \{y \in \mathbb{R}^n : \ |y-x| < r\}$.

(M_∞ — **the Hardy-Littlewood maximal function**.)

- Bojarski in 1991 showed that if $p \in (1, \infty)$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then for all $x,\ y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \lesssim |x-y| \left[M_{|x-y|}(\nabla f)(x) + M_{|x-y|}(\nabla f)(y)\right].$$

Sobolev Spaces — A Nice Observation (2) / I

Observe that $M_{|x-y|}(\nabla f) \in L^p(\mathbb{R}^n)$, since $\nabla f \in L^p(\mathbb{R}^n)$ and $M_{|x-y|}$ is bounded on $L^p(\mathbb{R}^n)$.

• **B. Bojarski**, Remarks on some geometric properties of Sobolev mappings, **Functional analysis & related topics (Sapporo, 1990), 65-76, World Sci. Publ., River Edge, NJ, 1991**.

• **Hajłasz** [H96] in 1996 proved that the **inverse** of Bojarski's observation is also true.

- [H96] **P. Hajłasz**, Sobolev spaces on an arbitrary metric space, **Potential Anal. 5 (1996), 403-415**.

Sobolev Spaces — A Nice Observation (3) / I

Theorem 1.1 Let $p \in (1, \infty)$ and f be a measurable function. Then $f \in \dot{W}^{1,p}(\mathbb{R}^n) \Longleftrightarrow$ there exists a $0 \le g \in L^p(\mathbb{R}^n)$ such that for a. e. $x,\, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \lesssim |x - y|[g(x) + g(y)].$$

Moreover, $|\nabla f| \lesssim g \ a.\,e.$

- This observation makes it possible to introduce the Sobolev space of **order 1** on an arbitrary metric space.

Hajłasz-Sobolev Spaces (1) / I

- $(\mathcal{X}, d, \mu)$: $\mathcal{X}$ — nonempty set; d — metric; μ — regular Borel measure

- $p \in (1, \infty)$, $s \in (0, 1]$

- The **homogeneous fractional Hajłasz-Sobolev space** $\dot{M}^{s,p}(\mathcal{X})$ is defined to be the set of all measurable functions $f \in L^p_{\mathrm{loc}}(\mathcal{X})$ for which there exists a $0 \le g \in L^p(\mathbb{R}^n)$ and a set $E \subset \mathcal{X}$ of measure zero such that for all $x,\ y \in \mathcal{X} \setminus E$,

$$(1.1) \qquad |f(x) - f(y)| \le [d(x, y)]^s [g(x) + g(y)].$$

- Denote by $\mathcal{D}(f)$ the class of all nonnegative Borel

Hajłasz-Sobolev Spaces (2) / I

functions g satisfying (1.1). Moreover, define

$$\|f\|_{\dot{M}^{s,p}(\mathcal{X})} \equiv \inf_{g\in\mathcal{D}(f)}\{\|g\|_{L^p(\mathbb{R}^n)}\}.$$

Let $M^{s,p}(\mathcal{X}) \equiv L^p(\mathcal{X}) \cap \dot{M}^{s,p}(\mathcal{X})$ and, for all $f \in M^{s,p}(\mathcal{X})$, let

$$\|f\|_{M^{s,p}(\mathcal{X})} \equiv \|f\|_{L^p(\mathcal{X})} + \|f\|_{\dot{M}^{s,p}(\mathcal{X})}.$$

Remarks:

- $\dot{M}^{1,p}(\mathcal{X})$ & $M^{1,p}(\mathcal{X})$ were introduced by Hajłasz [H96].
- $\dot{M}^{s,p}(\mathcal{X})$ & $M^{s,p}(\mathcal{X})$ when $s \in (0,1)$ were introduced by Hu [Hu03] for subsets (fractals) of $\mathbb{R}^n$ and Yang [Y03] for metric measure spaces.

Hajłasz-Sobolev Spaces (3) / I

▶ It was proved in [H96] that

$$\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n) = \dot{F}^{1}_{p,2}(\mathbb{R}^n)$$

and in [Y03] that when $s \in (0, 1)$,

$$\dot{M}^{s,p}(\mathbb{R}^n) = \dot{F}^{s}_{p,\infty}(\mathbb{R}^n).$$

• [Hu03] **J. Hu**, A note on Hajłasz-Sobolev spaces on fractals, **J. Math. Anal. Appl. 280 (2003), 91-101**.

• [Y03] **D. Yang**, New characterizations of Hajłasz-Sobolev spaces on metric spaces, **Sci. China Ser. A 46 (2003), 675-689**.

RD-Spaces (1) / I

- A triple $(\mathcal{X}, d, \mu)$: $\mathcal{X}$ is a non-empty set, d a quasi-metric (usually, for simplicity, metric), and μ a regular Borel measure.

▶ A space of homogenous type of Coifman-Weiss: if μ-**doubling** ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

▶ An RD-space, if μ is both doubling and **reverse-doubling** ($\mu(B(x, 2r)) \geq C_0 \mu(B(x, r))$ and $C_0 > 1$).

▶ There exist many examples of RD-spaces. Especially, all **connected** spaces of homogeneous type are RD-spaces.

RD-Spaces (2) / I

• [HMY08] **Y. Han, D. Müller and D. Yang**, A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces, **Abstr. Appl. Anal. 2008 Art. ID 893409, 250 pp**.

• **D. Yang and Y. Zhou**, New properties of Besov and Triebel-Lizorkin spaces on RD-spaces, **Manuscripta Math. 134 (2011), 59-90**.

► Notation: For any subset E of $\mathcal{X}$ and $f \in L^1_{\rm loc}(\mathcal{X})$, let

$$\fint_E f(x)\,d\mu(x) = \frac{1}{\mu(E)}\int_E f(x)\,d\mu(x).$$

Equivalent Characterizations of $\dot{W}^{1,p}(\mathbb{R}^n)$ (1) / I

- $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, $p \in (1, \infty)$

$$\Longleftrightarrow \|f\|_{\dot{L}_b(1,p)} \equiv$$

$$\sup_{k\in\mathbb{Z}} 2^k \left(\int_{\mathbb{R}^n} ⨍_{B(x,2^{-k})} |f(x) - f(y)|^p \, dy \, dx \right)^{1/p} < \infty;$$

([GKS10] (SHT))

$$\Longleftrightarrow \|f\|_{\dot{L}_t(1,p)} \equiv$$

$$\left\{ \int_{\mathbb{R}^n} \left(\sup_{k\in\mathbb{Z}} 2^k ⨍_{B(x,2^{-k})} |f(x) - f(y)| \, dy \right)^p dx \right\}^{1/p} < \infty;$$

([Y03] (SHT))

Equivalent Characterizations of $\dot{W}^{1,p}(\mathbb{R}^n)$ / (2)

$$\Longleftrightarrow \|f\|_{\widetilde{\dot{L}}_t(1,p)} \equiv$$

$$\left\{ \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} 2^{2k} \left| f(x) - \fint_{B(x,2^{-k})} f(y)\,dy \right|^2 \right)^{p/2} dx \right\}^{1/p} < \infty;$$

([W69], [AMV10])

$$\Longleftrightarrow f \in \dot{F}^1_{p,2}(\mathbb{R}^n).$$

Moreover,

$$\|f\|_{\dot{W}^{1,p}(\mathbb{R}^n)} \sim \|f\|_{\dot{L}_b(1,p)} \sim \|f\|_{\dot{L}_t(1,p)} \sim \|f\|_{\widetilde{\dot{L}}_t(1,p)} \sim \|f\|_{\dot{F}^1_{p,2}(\mathbb{R}^n)}.$$

Equivalent Characterizations of $\dot{W}^{1,p}(\mathbb{R}^n)$ (3) / I

• [GKS10] **A. Gogatishvili, P. Koskela & N. Shanmugalingam**, Interpolation properties of Besov spaces defined on metric spaces, **Math. Nachr. 283 (2010), 215-231**.

• [W69] **R. L. Wheeden**, Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type, **Studia Math. 32 (1969), 73-93**.

• [AMV10] **R. Alabern, J. Mateu & J. Verdera**, A new characterization of Sobolev spaces on $\mathbb{R}^n$, **arXiv: 1011.0667**.

Equivalent Characterizations of $\dot{M}^{s,p}(\mathcal{X})$ / I

- $f \in \dot{M}^{s,p}(\mathcal{X})$, $p \in (1,\infty)$

$\iff \|f\|_{\dot{L}_t(s,p)} \equiv$

$$\left\{\int_{\mathcal{X}} \left[\sup_{k\in\mathbb{Z}} 2^{ks} \fint_{B(x,2^{-k})} |f(x)-f(y)|\,d\mu(y)\right]^p d\mu(x)\right\}^{1/p} < \infty,$$

$s \in (0,1]$, $\mathcal{X}$ — SHT; ([Y03])

$\iff f \in \dot{F}^s_{p,\infty}(\mathcal{X})$, $s \in (0,1)$, $\mathcal{X}$ — RD space; ([Y03] & [MY09])

- [MY09] **D. Müller and D. Yang**, A difference characterization of Besov and Triebel-Lizorkin spaces on RD-spaces, **Forum Math. 21 (2009), 259-298**.

Grand Triebel-Lizorkin Spaces on $\mathbb{R}^n$ (1) / I

- For $m \in (n+1, \infty)$, let

$$(1.2) \qquad \mathcal{A} \equiv \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x)\, dx = 0, \right.$$

$$\left. \sup_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le 1} \sup_{x \in \mathbb{R}^n} (1+|x|)^m |\partial^\alpha \phi(x)| \le 1 \right\}.$$

- The **grand** Triebel-Lizorkin space $\mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p \in (0, \infty)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \equiv \left\| \sup_{k \in \mathbb{Z}} 2^{ks} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * f| \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\phi_t(x) \equiv \frac{1}{t^n} \phi(\frac{x}{t})$ for all $t > 0$ and $x \in \mathbb{R}^n$.

Grand Triebel-Lizorkin Spaces on $\mathbb{R}^n$ (2) / I

▶ Regard $\mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n)$ as a quotient space (an **equivalent class**: $f + \mathbb{C}$ for any $f \in \mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n)$).

▶ **Theorem 1.2** ([KYZ10]). (i) For $s \in (0,1)$ and $p \in (n/(n+s), \infty)$,

$$\mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n) = \dot{F}^s_{p,\infty}(\mathbb{R}^n).$$

(ii) For $s \in (0,1]$ and $p \in (n/(n+s), \infty)$,

$$\mathcal{A}\dot{F}^s_{p,\infty}(\mathbb{R}^n) = \dot{M}^{s,p}(\mathbb{R}^n).$$

▶**Remarks**: (i) Recall that $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n) \subsetneqq \dot{F}^1_{p,\infty}(\mathbb{R}^n)$.

(ii) Thm. 1.2 is also true for any **RD-space** $\mathcal{X}$.

Some Remarks (1) / I

• [KYZ10] **P. Koskela, D. Yang and Y. Zhou**, A characterization of Hajłasz-Sobolev and Triebel-Lizorkin spaces via grand Littlewood-Paley functions, **J. Funct. Anal. 258 (2010), 2637-2661**.

▶ **Remarks**. (i) The choice of $\mathcal{A}$ in the definition of $\mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})$ is very **subtle** and it depends only on the **first-order derivatives** of test functions, which makes it possible to generalize Theorem 1.2 to metric measure spaces. (**Lipschitz regularity**)

Some Remarks (2) / I

(ii) Auchser, Russ and Tchamitchian [ART05, Theorem 6] characterized the Hardy-Sobolev space $\dot{F}^1_{p,\,2}(\mathbb{R}^n)$ via a maximal function which is obtained by transferring the **gradient on f** to a **size condition** on the **divergence of the vectors** formed by certain test functions. However, this characterization still depends on the **derivatives**.

• [ART05] **P. Auscher, E. Russ and P. Tchamitchian**, Hardy Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$, **J. Funct. Anal. 218 (2005), 54-109**.

Some Remarks (3) / I

(iii) We also point out that Cho [C99] characterized Hardy-Sobolev spaces $\dot{H}^{k,p}(\mathbb{R}^n) = \dot{F}^k_{p,2}(\mathbb{R}^n)$ with $k \in \mathbb{N}$ via a nontangential maximal function by transferring the **derivatives** on the distribution to a fixed **specially chosen Schwartz function**; see Theorem I of [C99].

- [C99] **Y.-K. Cho**, Inequalities related to H^p smoothness of Sobolev type, **Integral Equations Operator Theory 35 (1999), 471-484**.

Some Remarks (4) / I

(iv) We finally remark that a continuous version of the grand Littlewood-Paley g-function $(\sum_{k\in\mathbb{Z}} \sup_{\phi\in\mathcal{A}} |\phi_{2^{-k}} * f|^2)^{1/2}$ with a different choice of $\mathcal{A}$ was used by Wilson [W07] to solve a conjecture of R. Fefferman and E. M. Stein on the weighted boundedness of the classical Littlewood-Paley S-function.

• [W07] **M. Wilson**, The intrinsic square function, **Rev. Mat. Ibero. 23 (2007), 771-791**.

Proof of Theorem 1.2 / I

- Tools:
  - Calderón reproducing formula
  - Sobolev embedding theorem
  - Lebesgue differential theorem
- Inhomogeneous versions
- RD-spaces (Given a new characterization for $\dot{M}^{s,p}(\mathcal{X})$ via the **grand Littlewood-Paley g-function**)

Grand Lusin-area Characterization (1) / I

•[JYY] **X. Jiang, D. Yang & W. Yuan**, The grand Lusin-area characterization of Hajłasz-Sobolev spaces and Triebel-Lizorkin spaces, **Submitted**.

► For $\mathcal{A}$ as in (1.2), any $f \in \mathcal{S}'(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, the **grand Lusin-area** function (or the **grand non-tangential maximal** function) $\mathcal{A}\dot{S}^s_{\infty,a}(f)$ is defined by

$$\mathcal{A}\dot{S}^s_{\infty,a}(f)(x) \equiv \sup_{k\in\mathbb{Z}} 2^{ks} \sup_{y\in B(x,a2^{-k})} \sup_{\phi\in\mathcal{A}} |\phi_{2^{-k}} * f(y)|.$$

► $\mathcal{A}\dot{S}^s_{\infty,a}(f)$ is the special case of the **grand Lusin-area** function $\mathcal{A}\dot{S}^s_{q,a}(f)$ when $q = \infty$.

Grand Lusin-area Characterization (2) / I

- **Theorem 1.3** ([JYY]). Let $s \in (0, 1]$, $p \in (n/(n+s), \infty)$, $a \in (0, \infty)$ and $\mathcal{A}$ be as in (2.1). Then $f \in \dot{M}^{s,p}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{A}\dot{S}^s_{\infty,a}(f) \in L^p(\mathbb{R}^n)$. Moreover,

$$\|\cdot\|_{\dot{M}^{s,p}(\mathbb{R}^n)} \sim \|\mathcal{A}\dot{S}^s_{\infty,a}(\cdot)\|_{L^p(\mathbb{R}^n)}.$$

- Thm. 1.3 is also true for any **RD-space** $\mathcal{X}$.

- Applications to **real interpolation**:

  - **X. Jiang, W. Yuan & D. Yang**, Real interpolation for grand Besov and Triebel-Lizorkin spaces on RD-spaces, **Ann. Acad. Sci. Fenn. Math. 36 (2011), 509-529**.

II. Pointwise characterizations of Besov and Triebel-Lizorkin spaces

Fractional s−Hajłasz Gradient / II

• **P. Koskela, D. Yang and Y. Zhou**, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, **Adv. Math. 226 (2011), 3579-3621**.

▶**Definition 2.1** Let $s \in (0, \infty)$ and u be a measurable function on $\mathcal{X}$. A sequence of nonnegative measurable functions, $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$, is called a **fractional s-Hajłasz gradient** of u if there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in \mathcal{X} \setminus E$ satisfying $2^{-k-1} \le d(x, y) < 2^{-k}$,

$$|u(x) - u(y)| \le [d(x, y)]^s [g_k(x) + g_k(y)]. \tag{2.1}$$

Denote by $\mathbb{D}^s(u)$ the collection of all fractional s-Hajłasz gradients of u.

$\dot{M}^s_{p,\,q}(\mathcal{X})$ & $\dot{N}^s_{p,\,q}(\mathcal{X})$ / II

- The **homogeneous Hajłasz-Triebel-Lizorkin space** $\dot{M}^s_{p,\,q}(\mathcal{X})$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^s_{p,\,q}(\mathcal{X})} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \left\|\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q}\right\|_{L^p(\mathcal{X})} < \infty.$$

- The **homogeneous Hajłasz-Besov space** $\dot{N}^s_{p,\,q}(\mathcal{X})$ is the space of all measurable functions u such that

$$\|u\|_{\dot{N}^s_{p,\,q}(\mathcal{X})} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \left\|\left\{\|g_j\|_{L^p(\mathcal{X})}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q} < \infty.$$

Theorem 2.1 / II

▶ **Theorem 2.1** Let $n \in \mathbb{N}$.

(i) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (n/(n+s), \infty]$, then $\dot{M}^s_{p,\,q}(\mathcal{X}) = \dot{F}^s_{p,\,q}(\mathcal{X})$.

(ii) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (0, \infty]$, then $\dot{N}^s_{p,\,q}(\mathcal{X}) = \dot{B}^s_{p,\,q}(\mathcal{X})$.

• More **optimal** characterizations (for example, **difference** or **sharp function**) of $\dot{M}^s_{p,\,q}(\mathcal{X})$ and $\dot{N}^s_{p,\,q}(\mathcal{X})$ were recently established in

- ▶ **A. Gogatishvili, P. Koskela & Y. Zhou**, Characterizations of Besov and Triebel-Lizorkin spaces on metric measure spaces, **Forum Math.** (to appear).

Quasiconformal Maps / II

- Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f: \mathcal{X} \to \mathcal{Y}$ be a **homeomorphism**. If there exists $H \in (0, \infty)$ such that for all $x \in \mathcal{X}$,

$$\limsup_{r \to 0} \frac{\sup\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \le r\}}{\inf\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \ge r\}} \le H,$$

then f is called **quasiconformal**.

- A function space is called **invariant** under quasiconformal mappings, if both f and f^{-1} induce a **bounded composition operator**.

Theorem 2.2 / II

- **Theorem 2.2** (i) Let $n \geq 2$, $s \in (0, 1)$ and $q \in (n/(n+s), \infty]$. Then $\dot{F}^s_{n/s,\, q}(\mathbb{R}^n)$ is invariant under quasiconformal mappings of $\mathbb{R}^n$.

  (ii) Let $n \geq 2$, $s \in (0, 1]$ and $q \in (0, \infty]$. Then $\dot{M}^s_{n/s,\, q}(\mathbb{R}^n)$ is invariant under quasiconformal mappings of $\mathbb{R}^n$.

- Thm. 2.2 is true for any **RD-space** $\mathcal{X}$.

- Main tool: Theorem 2.1.

Known results / II

▶ Partly known results: i) $\mathrm{BMO}(\mathbb{R}^n)$ is quasiconformally invariant by

- **H. M. Reimann**, Functions of bounded mean oscillation and quasiconformal mappings, **Comment. Math. Helv. 49 (1974), 260-276**.

ii) If $s \in (0,\, 1)$, then $\dot{F}^s_{n/s,\,n/s}(\mathbb{R}^n)$ is quasiconformally invariant by

- **M. Bourdon and H. Pajot**, Cohomologie l_p et espaces de Besov, **J. Reine Angew. Math. 558 (2003), 85-108**.

III. Pointwise characterizations of Newton-Besov and Newton-Triebel-Lizorkin spaces

p-Modulus (1) / III

- Let $\mathcal{X}$ be a **metric space endowed with a doubling measure**. A **curve** γ in $\mathcal{X}$ is a continuous mapping from an interval into $\mathcal{X}$.

- Let Γ be a **collection** of some curves in $\mathcal{X}$. A function ρ is called **admissible for the family** Γ if ρ is a nonnegative Borel measurable function satisfying that for **all rectifiable curves** γ **in** Γ, the path integral $\int_\gamma \rho\, ds \geq 1$.

- For all $p \in [1, \infty)$, the p**-modulus of** Γ, denoted by $\mathrm{Mod}_p(\Gamma)$, is defined to be the number

$$\mathrm{Mod}_p(\Gamma) \equiv \inf_\rho \|\rho\|^p_{L^p(\mathcal{X})},$$

where the **infimum** is taken over **all admissible functions** ρ for the family Γ.

p-Modulus (2) / III

- p-modulus is an **outer measure** on the family of all curves in $\mathcal{X}$, and the p-modulus of the set of all non-rectifiable curves in $\mathcal{X}$ is zero.

- [H01] **J. Heinonen**, Lectures on Analysis on Metric Spaces, **Springer-Verlag, New York, 2001**.

p-Weak Upper Gradient / III

Let u be a measurable function on $\mathcal{X}$. A Borel measurable non-negative function g on $\mathcal{X}$ is an **upper gradient** of u if

$$|u \circ \gamma(b) - u \circ \gamma(a)| \leq \int_\gamma g\, ds \tag{3.1}$$

holds for all non-constant compact rectifiable curves $\gamma : [a, b] \to \mathcal{X}$. Moreover, if (3.1) fails only on a family Γ of non-constant compact rectifiable curves with $\mathrm{Mod}_p(\Gamma) = 0$, then g is called a **p-weak upper gradient of u**.

- **J. Heinonen and P. Koskela**, Quasiconformal maps in metric spaces with controlled geometry, **Acta Math. 181 (1998), 1-61**.
- **P. Koskela and P. MacManus**, Quasiconformal mappings and Sobolev spaces, **Studia Math. 131 (1998), 1-17**.

Newton-Sobolev Spaces (1) / III

The **Newton-Sobolev space** $N^{1,p}(\mathcal{X})$ for $p \in [1, \infty)$ is defined to be the set of all measurable functions u on $\mathcal{X}$ such that $u \in L^p(\mathcal{X})$ and u **has a p-weak upper gradient** $g \in L^p(\mathcal{X})$.

The **norm** of $N^{1,p}(\mathcal{X})$ is defined by

$$\|u\|_{N^{1,p}(\mathcal{X})} \equiv \|u\|_{L^p(\mathcal{X})} + \inf_g \|g\|_{L^p(\mathcal{X})},$$

where the infimum is taken over **all p-weak upper gradients** g of u.

Newton-Sobolev Spaces (2) / III

- Advantage: **Strong locality**

If a function is **constant on a measurable set**, then we can take **upper gradient to be zero almost everywhere on that set**; however, we cannot take the **Hajłasz gradient to be zero almost everywhere on that set**.

So when it comes to study of PDEs such as **regularity theory**, the upper gradient version is more convenient than the Hajłasz gradient.

- **N. Shanmugalingam**, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, **Rev. Mat. Iberoamericana 16 (2000), 243-279**.

Upper-Gradients at Level k / III

Let $k \in \mathbb{Z}$ and $u : \mathcal{X} \to \mathbb{R}$, $g_k : \mathcal{X} \to [0, \infty]$ be such that u is measurable and g_k Borel measurable. The function g_k is called an **upper gradient of u at level k** if whenever $\gamma : [a, b] \to \mathcal{X}$ is a rectifiable curve with $2^{-k-1} \le \ell(\gamma) < 2^{-k}$, then

$$|u \circ \gamma(b) - u \circ \gamma(a)| \le \int_\gamma g_k \, ds. \tag{3.2}$$

For all $k \in \mathbb{Z}$, let Γ_k be the collection of all rectifiable curves $\gamma : \; [a, b] \to \mathcal{X}$ satisfying that $2^{-k-1} \le \ell(\gamma) < 2^{-k}$ and (3.2) fails. If $\mathrm{Mod}_p(\Gamma_k) = 0$, then g_k is called a **p-weak upper gradient of u at level k**.

$D_{\mathbb{N}}(u)$ / III

$D_{\mathbb{N}}(u)$ denotes the collection of all sequences $\{g_k\}_{k\in\mathbb{N}}$ such that each g_k is a p-weak upper gradient of u at level k.

- **N. Shanmugalingam, D. Yang & W. Yuan**, Newton-Besov spaces and Newton-Triebel-Lizorkin spaces, **in progress**.

Newton-Besov Spaces / III

Let $s \in [0,1]$, $p \in [1,\infty)$ and $q \in (0,\infty]$.

(i) The **Newton-Besov space** $NB^s_{p,q}(\mathcal{X})$ is defined to be the set of all measurable functions u on $\mathcal{X}$ such that $u \in L^p(\mathcal{X})$ and

$$\|u\|_{NB^s_{p,q}(\mathcal{X})} \equiv \|u\|_{L^p(\mathcal{X})} + \inf_{D_{\mathbb{N}}(u)\ni\{g_k\}} \left\| \left\{ 2^{-k(1-s)} \|g_k\|_{L^p(\mathcal{X})} \right\}_{k\in\mathbb{N}} \right\|_{\ell^q} < \infty$$

with the usual modification when $q = \infty$.

Newton-Triebel-Lizorkin Spaces / III

The **Newton-Triebel-Lizorkin space** $NF^s_{p,q}(\mathcal{X})$ is defined to be the set of all functions $u \in L^p(\mathcal{X})$ such that

$$\|u\|_{NF^s_{p,q}(\mathcal{X})} \equiv \|u\|_{L^p(\mathcal{X})} + \inf_{D_{\mathbb{N}}(u)\ni\{g_k\}} \left\| \left\| \left\{2^{-k(1-s)} g_k\right\}_{k\in\mathbb{N}} \right\|_{\ell^q} \right\|_{L^p(\mathcal{X})} < \infty$$

with the usual modification when $q = \infty$.

Some Properties / III

▶ **Proposition 3.1** For all $p \in (1, \infty)$, $NB^1_{p,\infty}(\mathcal{X}) = N^{1,p}(\mathcal{X})$.

▶ **Theorem 3.1** Let $s \in [0, 1]$, $p \in [1, \infty)$ and $q \in (0, \infty]$. The spaces $NB^s_{p,q}(\mathcal{X})$ and $NF^s_{p,q}(\mathcal{X})$ are **complete**. In particular, when $p \in [1, \infty)$ and $q \in [1, \infty]$, $NB^s_{p,q}(\mathcal{X})$ and $NF^s_{p,q}(\mathcal{X})$ are **Banach spaces**.

Semmes pencil of curves / III

A metric space $(\mathcal{X}, d)$ is called to have a **Semmes pencil of curves** if $\exists$ a positive constant C such that for all $x, y \in \mathcal{X}$ with $x \neq y$, there exists a family Γ_{xy} of **curves connecting** x **to** y **with length at most** $C\, d(x,y)$, and a **probability measure** α_{xy} **on** Γ_{xy} such that whenever $A \subset \mathcal{X}$ is a Borel measurable set,

$$\int_{\Gamma_{xy}} \mathcal{H}^1(\gamma \cap A)\, d\alpha_{xy}(\gamma) \le C \left\{ \int_A \frac{d(x,z)}{\mu(B(x, d(x,z)))}\, d\mu(z) + \int_A \frac{d(y,z)}{\mu(B(y, d(y,z)))}\, d\mu(z) \right\}.$$

► **Examples**: Euclidean space, Laakso space, the Bordon-Pajot space, and the Heisenberg group.

Relations / III

► **Theorem 3.2** Suppose that $\mathcal{X}$ supports **a Semmes pencil of curves** and μ is a doubling measure. ($\equiv\equiv\Rightarrow$ $\mathcal{X}$ is an RD-space.) Let $s \in (0, 1]$ and $p \in (1, \infty)$.

(i) If $q \in (0, \infty]$, then $NB^s_{p,q}(\mathcal{X}) \subsetneqq N^s_{p,q}(\mathcal{X})$ ($= B^s_{p,q}(\mathcal{X})$ when $s \in (0, 1)$).

(ii) If $q \in (1, \infty]$, then $NF^s_{p,q}(\mathcal{X}) \subsetneqq M^s_{p,q}(\mathcal{X})$ ($= F^s_{p,q}(\mathcal{X})$ when $s \in (0, 1)$).

IV. Further remarks

Further Remarks (1) / IV

- Motivated by [W69] and [AMV10], Yang-Yuan-Zhou proved

For $s \in (0,2)$ and $p,\ q \in (1,\infty]$,

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \sim \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \left| f - \fint_{B(\cdot,2^{-k})} f(y)\,dy \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

- **Method** is totally different from [W69] and [AMV10].

- **D. Yang, W. Yuan & Y. Zhou**, A new characterization of Triebel-Lizorkin spaces on $\mathbb{R}^n$, **Submitted**.

Further Remarks (2) / IV

- Sobolev spaces associated with **operators**

  - **L. Yan and D. Yang**, New Sobolev spaces via generalized Poincaré inequalities on metric measure spaces, **Math. Z. 255 (2007), 133-159**.
  - **S. Hofmann, S. Mayboroda & A. McIntosh**, Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces, **Ann. Sci. École Norm. Sup. (4) (to appear)** or **arXiv:1002.0792**.

Further Remarks (3) / IV

- Triebel [T10] introduced the **higher** version of Hajłasz-Sobolev spaces on $\mathbb{R}^n$ via **higher differences**, and some very interesting applications are given in [T11] and [HT11]:

- [HT11] **D. D. Haroske & H. Triebel**, Embeddings of function spaces: a criterion in terms of differences, **Complex Var. Elliptic Equ.** (to appear).
- [T10] **H. Triebel**, Sobolev-Besov spaces of measurable functions, **Studia Math. 201 (2010), 69-86.**
- [T11] **H. Triebel**, Limits of Besov norms, **Arch. Math. 96 (2011), 169-175.**

Thank you for your attention.