Haar bases in weighted Besov spaces

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Local Muckenhoupt weights

Definition (Rychkov,2001)

We define the weight class $\mathcal{A}_p^{loc}$ $(1 < p < \infty)$ to consist of all nonnegative locally integrable functions w defined on $\mathbb{R}^n$ for which

$$A_p^{loc}(w) := \sup_{|Q|\le 1} \frac{1}{|Q|^p} \int_Q w(x)dx \left(\int_Q w^{1-p'}(x)dx \right)^{p-1} < \infty \qquad (1)$$

and $\mathcal{A}_1^{loc}$

$$A_1^{loc}(w) = \sup_{|Q|\le 1} \frac{w(Q)}{|Q|} \left\| w^{-1} \right\|_{L_\infty(Q)} < \infty. \qquad (2)$$

$w \in \mathcal{A}_p \Rightarrow w \in \mathcal{A}_p^{loc}$

Properties

Lemma

Let $1 < p_1 < p_2 < \infty$. Then $\mathcal{A}_{p_1}^{loc} \subset \mathcal{A}_{p_2}^{loc} \subset \mathcal{A}_{\infty}^{loc}$.
Conversely, if $w \in \mathcal{A}_{\infty}^{loc}$, then $w \in \mathcal{A}_{p}^{loc}$ for some $p < \infty$.

We denote $r_w = \inf \left\{1 \leq p < \infty : w \in \mathcal{A}_p^{loc}\right\}$.
$\mathcal{A}_{\infty}^{loc} = \bigcup_{p \geq 1} \mathcal{A}_p^{loc}$

Example

An example of a weight, which is in $\mathcal{A}_p^{loc} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$.

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \leq 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases}$$

for $\alpha, \beta > -n$. For $\alpha < (p-1)n$ we have $w \in \mathcal{A}_p^{loc}$ and $r_w = \frac{\max(0,\alpha)}{n} + 1$. On the otherhand $\alpha, \beta < (p_1 - 1)n$ we have $w \in \mathcal{A}_{p_1}$ and $\widetilde{r_w} = \frac{\max(0,\alpha,\beta)}{n} + 1$. Taking β big enought we get that w is in $\mathcal{A}_p^{loc} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$ and $r_w < \widetilde{r_w}$.

Distribution class S'_e

We introduce a distribution class $\mathcal{S}'_e$, which is a topological dual to the following space:

$$\mathcal{S}_e := \{\psi \in C^\infty(\mathbb{R}^n) : q_N(\psi) < \infty \text{ for all } N \in \mathbb{N}\}, \tag{3}$$

where the seminorm

$$q_N(\psi) := \sup_{\alpha \in \mathbb{N}_0, |\alpha| \leq N} \left(\sup_{x \in \mathbb{R}^n} e^{N|x|} D^\alpha \psi(x) \right). \tag{4}$$

Besov spaces

Let $\varphi_0 \in \mathcal{D}$ be such that

$$\int_{\mathbb{R}^n} \varphi_0(x)\, dx \neq 0 \tag{5}$$

and

$$\int_{\mathbb{R}^n} x^\beta \varphi_0(x)\, dx = 0 \tag{6}$$

for some $\beta \in \mathbb{N}_0^n, 0 < |\beta| \leq B$. Let $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$ and $\varphi_j(x) = 2^{(j-1)n}\varphi(2^{j-1}x)$ for $j = 1, 2, \ldots$.
Then $\int_{\mathbb{R}^n} \varphi_j(x) x^\beta\, dx = 0$, if $|\beta| \leq B$. If $B = -1$ there is no vanishing moment condition.

Besov spaces

Definition (Rychkov,2001)

Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_\infty^{loc}$. Let function $\varphi_0 \in \mathcal{D}$ satisfy

$$\int_{\mathbb{R}^n} \varphi_0(x)dx \neq 0 \tag{7}$$

and

$$\int_{\mathbb{R}^n} x^\beta \varphi_0(x)dx = 0 \quad |\beta| < B, \tag{8}$$

where $B \geq [s]$. We define the weighted Besov spaces $B^{s,w}_{pq}(\mathbb{R}^n)$ as the sets of all $f \in \mathcal{S}'_e$ with finite norms

$$\left\| f | B^{s,w}_{pq}(\mathbb{R}^n) \right\|_{\varphi_0} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \varphi_j * f | L^w_p \right\|^q \right)^{1/q} < \infty. \tag{9}$$

The definition of the spaces $B^{s,w}_{pq}(\mathbb{R}^n)$ is independent of choice of the function φ_0.

Embeddings of unweighted spaces

Theorem

Let $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \le \infty$. *Then* $B^s_{pq}(\mathbb{R}^n) \hookrightarrow L^{loc}_{\max(1,p)}(\mathbb{R}^n)$ *if and only if*

$$s > n(\frac{1}{p} - 1)_+,\ 0 < p < \infty,\ 0 < q \le \infty \tag{10}$$

or

$$s = n(\frac{1}{p} - 1),\ 0 < p \le 1,\ 0 < q \le 1 \tag{11}$$

or

$$s = 0,\ 1 < p < \infty,\ 0 < q \le \min(p, 2). \tag{12}$$

Embeddings of unweighted spaces

Theorem

Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then the following assertions are equivalent:

1. $B^s_{pq}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$,
2. $B^s_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$,
3. *either* $s > \frac{n}{p}$ *or* $s = \frac{n}{p}$ *and* $0 < q \leq 1$.

Embeddings of spaces with Muckenhoupt weights

Proposition

Let w_1 and w_2 be two $\mathcal{A}_\infty^{loc}$ weights and let $-\infty < s_2 \leq s_1 < \infty$, $0 < p_1, p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$. We put

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ . \tag{13}$$

There is a continuous embedding $B^{s_1,w_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,w_2}_{p_2,q_2}(\mathbb{R}^n)$ if, and only if,

$$\left\{2^{-\nu(s_1-s_2)} \left\| \left\{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \right\}_m |\ell_{p^*} \right\| \right\}_\nu \in \ell_{q^*}. \tag{14}$$

Embeddings of spaces with Muckenhoupt weights

The embedding $B^{s_1,w_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,w_2}_{p_2,q_2}(\mathbb{R}^n)$ is compact if, and only if, (14) holds and, in addition,

$$\lim_{\nu\to\infty} 2^{-\nu(s_1-s_2)} \left\| \left\{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \right\}_m |\ell_{p^*} \right\| = 0 \quad \text{if } q^* = \infty, \tag{15}$$

and

$$\lim_{|m|\to\infty} (w_2(Q_{\nu,m}))^{-1/p_2} (w_1(Q_{\nu,m}))^{1/p_1} = \infty \quad \text{for all } \nu \in \mathbb{N}_0 \quad \text{if } p^* = \infty. \tag{16}$$

Embeddings of spaces with local Muckenhoupt weights

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty^{loc}$. *There is a continuous embedding* $B_{pq}^{s,w}(\mathbb{R}^n) \subset L_{\max(1,p)}^{loc}(\mathbb{R}^n)$ *if*

$$s > \frac{n}{p}\left(r_w - \min(1,p)\right), \; 0 < p, q \leq \infty. \tag{17}$$

For every $1 \leq \rho < \infty$ *there exists a weight* $w \in \mathcal{A}_\infty^{loc}$ *such that* $r_w = \rho$ *and if there is a continuous embedding* $B_{pq}^{s,w} \subset L_{\max(1,p)}^{loc}$, *then*

$$s \geq \frac{n}{p}(r_w - \min(1,p)) \quad \textit{if} \quad \begin{cases} 0 < p \leq \infty \textit{ and } 0 < q \leq 1, \\ 1 < p < \infty \textit{ and } 1 < q < \infty \end{cases} \tag{18}$$

and

$$s > \frac{n}{p}(r_w - \min(1,p)) \quad \textit{if} \quad \begin{cases} 0 < p \leq 1 \textit{ and } 1 < q < \infty, \\ 0 < p < \infty \textit{ and } q = \infty. \end{cases} \tag{19}$$

Embeddings of spaces with local Muckenhoupt weights

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_1^{loc}$. There is a continuous embedding $B^{s,w}_{pq}(\mathbb{R}^n) \subset L^{loc}_{\max(1,p)}(\mathbb{R}^n)$ if

$$s > \frac{n}{p}\left(1 - \min(1,p)\right), \; 0 < p < \infty, \; 0 < q \leq \infty \tag{20}$$

or

$$s = \frac{n}{p}\left(1 - p\right), \; 0 < p \leq 1, \; 0 < q \leq 1 \tag{21}$$

or

$$s = 0, \; 1 < p < \infty, \; 0 < q \leq \min(p, 2). \tag{22}$$

There exists an $\mathcal{A}_1^{loc}$ weight w such that if there is a continuous embedding $B^{s,w}_{pq}(\mathbb{R}^n) \subset L^{loc}_{\max(1,p)}(\mathbb{R}^n)$ then the above conditions are fulfilled.

Embeddings of spaces with local Muckenhoupt weights

Corollary

Let $1 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. For every $1 \leq \rho < \infty$ there exists a weight $w \in \mathcal{A}_\infty^{loc}$ such that $r_w = \rho$ and if there is no embedding $B_{pq}^{s,w}(\mathbb{R}^n) \subset L_1^{loc}(\mathbb{R}^n)$ then $s < \frac{n}{p}(r_w - p)_+$.

Theorem

1. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty^{loc}$. *There is a continuous embedding* $B^{s,w}_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, *if*

$$s > \frac{n}{p} r_w. \tag{23}$$

For every $1 \le \rho < \infty$ *there exists a weight* $w \in \mathcal{A}_\infty^{loc}$ *such that* $r_w = \rho$ *and if* $s \le \frac{n}{p} r_w$ *and* $1 < q \le \infty$ *then there is no embedding* $B^{s,w}_{pq}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, *or if* $s < \frac{n}{p} r_w$ *and* $0 < q \le \infty$ *then there is no embedding* $B^{s,w}_{pq}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$.

2. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_1^{loc}$. *There is a continuous embedding* $B^{s,w}_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, *if*

$$s > \frac{n}{p} \qquad \text{or} \qquad s = \frac{n}{p}, \quad 0 < q \le 1. \tag{24}$$

There exists an $\mathcal{A}_1^{loc}$ *weight* w *such that if there is a continuous embedding* $B^{s,w}_{pq}(\mathbb{R}^n) \subset L_\infty^{loc}(\mathbb{R}^n)$ *then the above conditions are fulfilled.*

Local means

Definition (Triebel)

Let $A, B \in \mathbb{N}_0$ and $C > 0$. Then C^A-functions $k_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called *kernels* of local means if

$$\text{supp } k_{jm} \subset CQ_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{25}$$

there exist all (classical) derivatives $D^\alpha k_{jm} \in C(\mathbb{R}^n)$ with $|\alpha| \leq A$ such that

$$|D^\alpha k_{jm}(x)| \leq 2^{jn+j|\alpha|}, \quad |\alpha| \leq A, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{26}$$

and

$$\int_{\mathbb{R}^n} x^\beta k_{jm}(x) dx = 0, \quad |\beta| < B, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{27}$$

Local means

Definition

Let $f \in D'_A(\mathbb{R}^n)$. Let k_{jm} be kernels according to Definition 9. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y) f(y) dy, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{28}$$

are *local means*. Furthermore,

$$k(f) = \{k_{jm}(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{29}$$

Local means

Definition

Let $s \in \mathbb{R}$, $0 \leq p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}_\infty^{loc}$. Then $\bar{b}_{p,q}^{s,w}$ is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\} \tag{30}$$

such that

$$\left\|\lambda|\bar{b}_{p,q}^{s,w}\right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\int_{\mathbb{R}^n} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{jm}\chi_{jm}^{(p)}(x)\right|^p w(x)dx\right)^{q/p}\right)^{1/q} < \infty. \tag{31}$$

Local means

For $w \in \mathcal{A}_\infty^{loc}$ let us define

$$\sigma_p(w) = n\left(\frac{r_w}{\min(p, r_w)} - 1\right) + (r_w - 1)n.$$

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Assume that $w \in \mathcal{A}_\infty^{loc}$. Let k_{jm} be kernels according to Definition 9, where $A, B \in \mathbb{N}_0$ with

$$A \geq \max(0, [-s + \sigma_p(w)], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1), \quad B \geq \max(0, [s] + 1),$$

Let $C > 0$ be fixed. Let $k(f)$ be as in (28) and (29). Then for some $c > 0$ and all $f \in B^{s,w}_{p,q}(\mathbb{R}^n)$,

$$\left\| k(f) | \bar{b}^{s,w}_{p,q} \right\| \leq c \left\| f | B^{s,w}_{p,q}(\mathbb{R}^n) \right\| . \tag{32}$$

Multidimentional Haar wavelets

Let

$$h^M(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ -1 & \text{if } 1/2 \leq x < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

$$h^F(x) = |h^M(x)| \tag{34}$$

and

$$h^M_{jm} = h^M(2^j x - m), \quad j \in \mathbb{N}_0, m \in \mathbb{Z}, \tag{35}$$

be the orthogonal Haar basis in $L_2(\mathbb{R})$.
Haar wavelets on $\mathbb{R}^n$ we obtain by usual tensor product procedure

$$H^G_{jm} = 2^{jn/2} \prod_{r=1}^{n} h^{G_r}(2^j x_r - m_r), \tag{36}$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $G = (G_1, \dots, G_n) \in G^j$ and $G^0 = \{F, M\}^n$ and for $j > 0$ $G^j = \{F, M\}^{n*}$, where * indicates that at least one G_r must be an M.

$$\left\{ H^G_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j \right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Haar wavelets as local means

If we take $A = 0$ and $B = 1$ in Theorem 12 and define local means on regular distributions, Haar wavelets can serve as local means.

Sequence spaces

Definition

Let $s \in \mathbb{R}$, $0 \leq p < \infty$, $0 \leq q \leq \infty$ and $w \in \mathcal{A}_\infty^{loc}$. Then $b_{p,q}^{s,w}$ is the collection of all sequences

$$\lambda = \left\{ \lambda_{jm}^{G} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ G \in G^j \right\} \tag{37}$$

such that

$$\left\| \lambda | b_{p,q}^{s,w} \right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \left(\int_{\mathbb{R}^n} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{G} \chi_{jm}^{(p)}(x) \right|^p w(x) dx \right)^{q/p} \right)^{1/q} < \infty. \tag{38}$$

Daubechies wavelet bases

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}$. For Daubechies wavelets from $C^k(\mathbb{R}^n)$ we take

$$k \geq \max(0, [s] + 1, [\frac{nr_w}{p} - \frac{n}{p} - s] + 1, [\sigma_p(w) - s]). \tag{39}$$

Let $f \in S_e'(\mathbb{R}^n)$. Then $f \in B_{pq}^{s,w}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j,G,m} \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G, \tag{40}$$

where $\lambda \in b_{pq}^{s,w}$ and convergence in $S_e'(\mathbb{R}^n)$.

Daubechies wavelet bases

This representation is unique with

$$\lambda_{jm}^{G} = 2^{jn/2} \left(f, \Psi_{jm}^{G} \right) \tag{41}$$

and

$$I : f \mapsto \{2^{jn/2}(f, \Psi_{jm}^{G})\} \tag{42}$$

is a linear isomorphism of $B_{pq}^{s,w}(\mathbb{R}^n)$ onto $b_{pq}^{s,w}$.
If $0 \leq p, q < \infty$ then the system $\left\{\Psi_{jm}^{G}\right\}_{j,m,G}$ is an unconditional basis in $B_{pq}^{s,w}(\mathbb{R}^n)$.

Regularity conditions of weights

Definition

Let $w \in \mathcal{A}_\infty^{loc}$. For every $k, j \in \mathbb{N}_0$, $k > j$ and some constant $D > 1$ there exists $d > 0$ such that

$$\frac{w(Q_{kl})}{|Q_{kl}|} \leq d \frac{w(Q_{jm})}{|Q_{jm}|} \tag{43}$$

where $DQ_{kl} \cap Q_{jm} \neq \emptyset$ and $DQ_{kl} \not\subseteq Q_{jm}$. Such a condition we call *regularity condition.*

1.Polynomial weights

$$w(x) = |x|^{\alpha}, \quad -n < \alpha < n(p-1),\ p > 1, \tag{44}$$

then $w \in \mathcal{A}_p^{loc}$.
2.Logarithmic weights

$$w(x) = |x|^{\alpha} \log^{-\beta}(2 + |x|), \quad -n < \alpha < n(p-1),\ p > 1,\ \beta \in \mathbb{R}, \tag{45}$$

then $w \in \mathcal{A}_p^{loc}$.
3.

$$w(x) = \begin{cases} |x|^{\alpha}, & |x| \le 1 \\ |x|^{\beta}, & |x| > 1. \end{cases} \tag{46}$$

and $\alpha, \beta > -n$, $\alpha < n(p-1)$, $p > 1$, then $w \in \mathcal{A}_p^{loc}$.
4.Example of a weight, that doesn't satisfy the regularity condition:

$$w(x) = \begin{cases} 1, & x_1 \le 0, \\ |x|^{\alpha}, & \text{otherwise}. \end{cases} \tag{47}$$

$w \in \mathcal{A}_1^{loc}$ for $-n < \alpha \le 0$.
This weight doesn't satisfy the regularity condition for $\alpha < 0$.

Characteristic functions

Proposition

Let $0 < p, q < \infty$, $s \in \mathbb{R}$. *Let* $w \in \mathcal{A}_\infty^{loc}$ *satisfy regularity condition . If*

$$\frac{n}{p}(r_w - \min(1,p)) < s < \frac{1}{p}, \tag{48}$$

then $f = \sum_{jm} \mu_{jm}\chi_{jm}$, $\mu \in \bar{b}^{s,w}_{pq}$ *belongs to* $B^{s,w}_{pq}(\mathbb{R}^n)$ *and*

$$||f|B^{s,w}_{pq}(\mathbb{R}^n)|| \leq c||\mu|\bar{b}^{s,w}_{pq}|| \tag{49}$$

for some $c > 0$ *and all* $\mu \in \bar{b}^{s,w}_{pq}$.

Haar wavelet bases

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* $w \in \mathcal{A}_\infty^{loc}$ *satisfies regularity condition . Let*

$$\max(\frac{n}{p}(r_w - \min(1,p)), \sigma_p(w) - 1) < s < \min(1, \frac{1}{p}).$$

Let $f \in \mathcal{S}_e'(\mathbb{R}^n)$. *Then* $f \in B_{pq}^{s,w}(\mathbb{R}^n)$ *if and only if it can be represented as*

$$f = \sum_{j,G,m} \lambda_{jm}^G 2^{-jn/2} H_{jm}^G,$$

where $\lambda \in b_{pq}^{s,w}$ *and the series converges in* $S_e'(\mathbb{R}^n)$.

Haar wavelet bases

This representation is unique with

$$\lambda_{jm}^{G} = 2^{jn/2}\left(f, H_{jm}^{G}\right)$$

and

$$I : f \mapsto \{2^{jn/2}(f, H_{jm}^{G})\}$$

is a linear isomorphism of $B_{pq}^{s,w}(\mathbb{R}^n)$ onto $b_{pq}^{s,w}$.
If $0 < p, q < \infty$ then the system $\left\{H_{jm}^{G}\right\}_{j,m,G}$ is an unconditional basis in $B_{pq}^{s,w}(\mathbb{R}^n)$.