Besov and Triebel-Lizorkin spaces of variable smoothness and integrability

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Outline

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  - Isotropic vs. anisotropic spaces
- Recent and new function spaces
  - Function spaces of variable integrability
  - Function spaces of variable smoothness
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- Properties of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$
  - Traces
  - Embeddings
  - Local means
  - Differences
  - Decompositions
- Outlook

History and motivation

Classical spaces of functions:
Lebesgue spaces (L_p)
Continuous and continuously differentiable functions (C, C^k)
Hardy spaces (H_p)

Classical spaces of distributions:
Sobolev spaces

$$W_p^k(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \|D^\alpha f\|_p < \infty,\ |\alpha| \le k\}$$

1950's and 1960's: Slobodecki spaces, Besov spaces defined by differences, Bessel potential spaces

Typical properties:

Sobolev embedding theorem:

$$W_{p_0}^{k_0}(\mathbb{R}^n) \hookrightarrow W_{p_1}^{k_1}(\mathbb{R}^n)$$

if $0 \leq k_1 \leq k_0$ are natural numbers, $1 \leq p_0 \leq p_1 < \infty$ and

$$k_0 - \frac{n}{p_0} = k_1 - \frac{n}{p_1}$$

Trace embedding theorem: $(\operatorname{tr} f)(x') = f(x', 0)$

$$\operatorname{tr} : W_p^1(\mathbb{R}^n) \to W_p^{1-1/p}(\mathbb{R}^{n-1}), \quad 1 < p < \infty$$

...and many others...

Fourier-analytic function spaces

Besov and Triebel-Lizorkin spaces:
smooth dyadic resolution of unity
$\varphi \in S(\mathbb{R}^n) : \varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}$

$$\varphi_0 := \varphi, \quad \varphi_j(\cdot) := \varphi(2^{-j}\cdot) - \varphi(2^{-j+1}\cdot)$$

$$\sum_{j=0}^{\infty} \varphi_j = 1$$

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})^{\vee}, \quad \text{convergence in } S'(\mathbb{R}^n)$$

Definition:

(i) $s \in \mathbb{R}$, $0 < p, q \leq \infty$

$$\|f|B^s_{pq}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}^n)\|^q\right)^{1/q}$$

(ii) $s \in \mathbb{R}, 0 < p < \infty$, $0 < q \leq \infty$

$$\|f|F^s_{pq}(\mathbb{R}^n)\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\|$$

Advantages:

- Two (closely related) scales including many special cases ($W_p^k, 1 < p < \infty$, Zygmund spaces, Slobodecki spaces)
- Many tools and results available (like embeddings, traces, Fourier multipliers, pointwise multipliers, . . .)
- Many equivalent characterisations (differences, local means,...)
- Good decomposition properties (atoms, molecules, wavelets)
- Applications to PDE's, stochastic, numerics, . . .
- Many generalisations (anisotropic spaces, spaces with dominating mixed smoothness, modulation spaces, . . .)

Disadvantages:

- Rather complicated definition (three parameters, decomposition of unity, Fourier transform)
- Some important spaces are not included (L_1, L_∞, W_1^1, BV)
- Spaces on domains?

Isotropy vs. anisotropy

B and F spaces are isotropic
. . . i.e. invariant under shifts and rotations

sometimes inconvenient $\rightarrow$ anisotropic spaces, weighted spaces, spaces of dominating mixed smoothness, . . .

$h - p$ Finite Elements Method (Babuška,. . .)
piecewise analytic functions

Recent function spaces

Function spaces with variable integrability

$p : \mathbb{R}^n \to (0, \infty]$ - measurable function

$L_{p(\cdot)}(\mathbb{R}^n)$: all $f : \mathbb{R}^n \to [-\infty, \infty]$, such that there is a $\lambda > 0$

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \varphi_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) dx < \infty$$

is finite, where

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1 \end{cases}$$

...the Minkowski functional of $\{f : \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq 1\}$...

$\mathbb{R}^n_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\}$ and $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \mathbb{R}^n_\infty$

$$\|f|L_{p(\cdot)}(\mathbb{R}^n)\| = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}$$
$$= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n_0} \left(\frac{f(x)}{\lambda}\right)^{p(x)} dx < 1 \text{ and } |f(x)| < \lambda \text{ for a.e. } x \in \mathbb{R}^n_\infty\right\}$$

Norm if $p(\cdot) \geq 1$; a quasi-norm if $p^- := \inf_{z\in\mathbb{R}^n} p(z) > 0$

$$W^1_{p(\cdot)}(\mathbb{R}^n) = \{f \in L_{p(\cdot)}(\mathbb{R}^n) : \nabla f \in L_{p(\cdot)}(\mathbb{R}^n)\}$$

Orlicz (1931), Kováčik & Rákosník (1991)
Diening & Růžička ($\approx$ 2000)

Maximal operator in $L_{p(\cdot)}(\mathbb{R}^n)$ & regularity conditions

Definition:(Regularity assumptions): $g \in C(\mathbb{R}^d)$
(i) g is *locally* log-*Hölder continuous* ($g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$)

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n$$

(ii) g is *globally* log-*Hölder continuous* ($g \in C^{\log}(\mathbb{R}^n)$) if it is locally log-Hölder continuous and

$$\exists c > 0 \text{ and } g_\infty \in \mathbb{R} : |g(x) - g_\infty| \le \frac{c}{\log(e + |x|)}, \quad x \in \mathbb{R}^n$$

If $1/p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p^- > 1$, then M is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$

...recent book *Lebesgue and Sobolev spaces with variable exponents* by L. Diening, P. Hästo, P. Harjulehto, and M. Růžička.

Function spaces with generalised smoothness

Many different approaches, starting already in 1960's

Spaces with generalised smoothness:
Gol'dman, Kalyabin, Leopold, Farkas, Moura, ...

Replace 2^{js} by σ_j

Spaces of variable smoothness:
Unterberger, Leopold, Besov, Almeida, Samko, ...

Replace s by $s(x)$

2-microlocal spaces:
Peetre, Bony, Jaffard, Kempka, ...

Replace 2^{js} by $w_j(x)$

New function spaces: Variable smoothness AND integrability

$$0 < p^- := \inf_{z\in\mathbb{R}^n} p(z) \le p(x) \le \sup_{z\in\mathbb{R}^n} p(z) =: p^+ < \infty, \quad x \in \mathbb{R}^n$$

Definition of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$
(L. Diening, P. Hästö, and S. Roudenko, 2009)
$s : \mathbb{R}^n \to \mathbb{R}$, $p, q : \mathbb{R}^n \to (0, \infty]$ - measurable functions

$$\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| = \left\| \left(\sum_{j=0}^{\infty} 2^{js(\cdot)q(\cdot)} |(\varphi_j \widehat{f})^\vee(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} |L_{p(\cdot)}(\mathbb{R}^n) \right\|$$

$$\|f|B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} \|2^{js(\cdot)} (\varphi_j \widehat{f})^\vee(\cdot)|L_{p(\cdot)}(\mathbb{R}^n)\|^q \right)^{1/q}$$

Definition of $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ (A. Almeida and P. Hästö, 2010)

$$\|f|B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| = \|2^{js(\cdot)}(\varphi_j\widehat{f})^\vee(\cdot)|\ell_{q(\cdot)}(L_{p(\cdot)})\|$$

$(f_\nu)_{\nu\in\mathbb{N}_0}$ ... sequence of $L_{p(\cdot)}(\mathbb{R}^n)$ functions

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \le 1\right\}$$

If $q^+ < \infty$:

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_\nu \||f_\nu|^{q(\cdot)}|L_{\frac{p(\cdot)}{q(\cdot)}}\|$$

The (quasi-)norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usual by

$$\|f_\nu|\ell_{q(\cdot)}(L_{p(\cdot)})\| = \inf\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \le 1\}$$

Properties of $A^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

Hardy-Littlewood maximal operator M is not bounded on $L_{p(\cdot)}(\ell_{q(\cdot)})$ and $\ell_{q(\cdot)}(L_{p(\cdot)})$ for $q(\cdot)$ variable!

Instead - convolutions with radial decaying kernels!

$$\eta_{\nu,m}(x) := 2^{n\nu}(1+2^{\nu}|x|)^{-m}$$

Theorem:(i) (DHR09) For $m > n$ and $1/p, 1/q \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$:

$$\|(\eta_{\nu,m} * f_\nu)_{\nu\in\mathbb{N}_0}|L_{p(\cdot)}(\ell_{q(\cdot)})\| \lesssim \|(f_\nu)_{\nu\in\mathbb{N}_0}|L_{p(\cdot)}(\ell_{q(\cdot)})\|$$

(ii) (AH10 & KV11) For $m > n + c_{\log}(1/q)$ and $1/p, 1/q \in C^{\log}(\mathbb{R}^n)$ with $p(\cdot) \geq 1$:

$$\|(\eta_{\nu,m} * f_\nu)_{\nu\in\mathbb{N}_0}|\ell_{q(\cdot)}(L_{p(\cdot)})\| \lesssim \|(f_\nu)_{\nu\in\mathbb{N}_0}|\ell_{q(\cdot)}(L_{p(\cdot)})\|$$

r-trick:

Theorem: Let $r > 0$, $\nu \geq 0$ and $m > n$. Then

$$|g(x)| \leq c(r,m,n)(\eta_{\nu,m} * |g|^r)^{1/r}(x)$$

for all $g \in S'(\mathbb{R}^d)$ with $\operatorname{supp}\hat{g} \subset \{\xi : |\xi| \leq 2^{\nu+1}\}$.

Properties of the new function spaces

- Independence on the decomposition of unity
- Boundedness of the φ-transform (... Frazier & Jawerth ...)
- Atomic & molecular decomposition
- Wavelet decomposition
- **Traces**
- **Sobolev embeddings**
- **Local means**
- **Differences**
- Fourier multipliers

Traces

Traces on hyperplanes

Theorem(DHR): Under regularity conditions on p, q and s and if

$$s(\cdot) - \frac{1}{p(\cdot)} - (n-1)\left(\frac{1}{\min(p(\cdot),1)} - 1\right) > 0$$

then

$$\operatorname{tr} F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1})$$

Sobolev Embeddings

(V09, AH10)

$$s_1(x) \leq s_0(x), \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n$$

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

$$B^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) > 0 \implies F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$$

Local means

Definition of B- and F-spaces works with $\varphi_j^\vee * f$

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \|2^{js}(\varphi_j^\vee * f)|\ell_q(L_p)\|$$

Under several conditions (smoothness, vanishing moments, Tauberian conditions) this may be replaced by $\psi_j * f$

Especially, ψ_j may have compact support → *local means*

Connected with the boundedness of the *Peetre maximal operator*

Both these techniques extended to $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in (KV11)

Characterisation by differences

First order differences:

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad x \in \mathbb{R}^n$$

Higher order differences:

$$\Delta_h^M f(x) = \Delta_h^1(\Delta_h^{M-1} f)(x), \quad M = 2, 3, \ldots$$

Ball means of differences

$$d_t^M f(x) = t^{-n} \int_{|h| \leq t} |\Delta_h^M f(x)| dh = \int_B |\Delta_{th}^M f(x)| dh,$$

$B = \{y \in \mathbb{R}^n : |y| < 1\}$, $t > 0$ and $M \in \mathbb{N}$

$$\sigma_p := n\left(\frac{1}{\min(p,1)} - 1\right) \quad \text{and} \quad \sigma_{p,q} := n\left(\frac{1}{\min(p,q,1)} - 1\right)$$

Theorem: (cf. Triebel, *Theory of function spaces*; many forerunners: Nikol'skij, Lizorkin, Stein, Strichartz, Kalyabin, Besov, ...)
(i) Let $0 < p < \infty$, $0 < q \leq \infty$ and $\sigma_{pq} < s < M$. Then

$$\|f|F^s_{p,q}(\mathbb{R}^n)\|^* := \|f\|_p + \left\| \left(\int_0^\infty t^{-sq} \left(d_t^M f(x) \right)^q \frac{dt}{t} \right)^{1/q} \middle| L_p(\mathbb{R}^n) \right\|$$

is an equivalent quasinorm on $F^s_{p,q}(\mathbb{R}^n)$.
(ii) Let $0 < p, q \leq \infty$ and $\sigma_p < s < M$. Then

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|^{**} := \|f|L_p(\mathbb{R}^n)\| + \left\| \left(2^{ks} d^M_{2^{-k}} f(x) \right)_{k=-\infty}^{\infty} \middle| \ell_q(L_p) \right\|$$

is an equivalent quasinorm on $B^s_{p,q}(\mathbb{R}^n)$.

Triebel-Lizorkin spaces

$$\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^* := \|f\|_{p(\cdot)} + \left\| \left(\int_0^\infty t^{-s(x)q(x)} \left(d_t^M f(x) \right)^{q(x)} \frac{dt}{t} \right)^{1/q(x)} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|$$

Discretized counterpart

$$\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^{**} := \|f\|_{p(\cdot)} + \left\| \left(2^{ks(x)} d^M_{2^{-k}} f(x) \right)_{k=-\infty}^{\infty} \middle| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

Besov spaces

$$\|f|B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^{**} := \|f\|_{p(\cdot)} + \left\| \left(2^{ks(x)} d^M_{2^{-k}} f(x) \right)_{k=-\infty}^{\infty} \middle| \ell_{q(\cdot)}(L_{p(\cdot)}) \right\|$$

Theorem (KV11): Let $1/p, 1/q \in C^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$, $s \in C^{\log}_{loc}(\mathbb{R}^n)$, $M \in \mathbb{N}$ with $M > s^+$ and

$$s^- > \sigma_{p^-,q^-} \cdot \left[1 + \frac{c_{\log}(s)}{n} \cdot \min(p^-, q^-)\right]$$

Then

$$F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \{f \in L_{p(\cdot)}(\mathbb{R}^n) \cap S'(\mathbb{R}^n) : \|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^* < \infty\}$$

and $\|\cdot|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$ and $\|\cdot|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^*$ are equivalent on $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. The same holds true for $\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^{**}$.

Theorem (KV11): Let $1/p, 1/q \in C^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{loc}(\mathbb{R}^n)$, $M \in \mathbb{N}$ with $M > s^+$ and

$$s^- > \sigma_{p^-} \cdot \left[1 + \frac{c_{\log}(1/q)}{n} + \frac{c_{\log}(s)}{n} \cdot p^-\right]$$

Then

$$B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \{f \in L_{p(\cdot)}(\mathbb{R}^n) \cap S'(\mathbb{R}^n) : \|f|B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^{**} < \infty\}$$

and $\|\cdot|B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$ and $\|\cdot|B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^{**}$ are equivalent on $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Decomposition techniques

Sequence spaces

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$: $Q_{j,m}$ cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-j}m$ and with side length 2^{-j}

$\chi_{j,m} := \chi_{Q_{j,m}}$ its characteristic function

$$\lambda = \{\lambda_{j,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

$$\|\lambda | f^{s(\cdot)}_{p(\cdot),q(\cdot)}\| := \left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |2^{js(\cdot)} \lambda_{j,m} \chi_{j,m}(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} | L_{p(\cdot)}(\mathbb{R}^n) \right\|$$

$$= \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} |\lambda_{j,m}| \chi_{j,m}(\cdot) | L_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

$$\|\lambda | b^{s(\cdot)}_{p(\cdot),q(\cdot)}\| := \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} |\lambda_{j,m}| \chi_{j,m}(\cdot) | \ell_{q(\cdot)}(L_{p(\cdot)}) \right\|$$

φ-transform: $S_\varphi f = \big(\langle f, \varphi_{j,m}\rangle\big)_{j,m}$
$\varphi_{j,m}\ldots$ shifts and dilations of basic function(s) φ

Theorem: (DHR09)

$$S_\varphi : F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \to f^{s(\cdot)}_{p(\cdot),q(\cdot)}$$

Theorem: (K10): Atomic, molecular and wavelet decomposition of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q}$ (and of 2-microlocal variants of these)

open for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

Outlook:

Spaces on domains:

- rather difficult due to the use of Fourier transform in the definition
- usually defined by restriction
- characterisation by differences
- traces (at least for C^∞-domains)
- extension operators (at least for C^∞-domains)
- wavelets on domains
- rough domains: Lipschitz?, or with boundary described again in the terms of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$-spaces?

Duality

Mapping properties of classical operators

Approximation theory

Theoretical concepts of numerical analysis:

- widths
- entropy numbers
- best k-term approximation (through sequence spaces)
- linear vs. non-linear approximation
- greedy algorithms?

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