Kolmogorov widths of weighted Sobolev and Besov classes

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Let X be a linear space with a seminorm $\|\cdot\| : X \to \mathbb{R}_+ \cup \{+\infty\}$, $n \in \mathbb{Z}_+$,

$$\mathcal{L}_n(X) = \{L \subset X : \ L \text{ is a subspace, } \dim L \le n\}.$$

Let M be a subset of X. The Kolmogorov n-width of M is defined by a formula

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|_X.$$

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Let

$$\|x\|_{l_q^n} = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q}, \quad x = (x_1, \dots, x_n), \quad l_q^n = (\mathbb{R}^n, \|\cdot\|_{l_q^n}),$$

$$B_p^n = \{x \in l_p^n : \|x\|_{l_p^n} \le 1\}.$$

Some classical results

- A. Pietsch (1974), M.I. Stesin (1975):

$$d_n(B_p^m, l_q^m) = (m-n)^{\frac{1}{q}-\frac{1}{p}}, \ \ 1 \le q \le p \le \infty;$$

- B.S. Kashin, E.D. Gluskin (1977–1981):

$$d_n(B_p^m, l_q^m) \underset{p,q}{\asymp} \Phi(n, m, p, q),$$

where $\Phi(n, m, p, q)$ is

$$\min\left\{1, \left(m^{1/q}n^{-1/2}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)}\right\}, \ 2 \le p \le q < \infty,$$

$$\max\left\{m^{\frac{1}{q}-\frac{1}{p}}, \min\left(1, m^{\frac{1}{q}}n^{-\frac{1}{2}}\right)\left(1-\frac{n}{m}\right)^{1/2}\right\}, \ 1 \le p < 2 \le q < \infty,$$

$$\max\left\{m^{\frac{1}{q}-\frac{1}{p}}, \left(1-\frac{n}{m}\right)^{\left(\frac{1}{q}-\frac{1}{p}\right)/\left(1-\frac{2}{p}\right)}\right\}, \ 1 \le p \le q \le 2.$$

Let $\Omega \subset \mathbb{R}^d$ be a domain, $d \in \mathbb{N}$, $1 \le p$, $q < \infty$, g, $v : \Omega \to \mathbb{R}_+$ be measurable functions,

$$\varphi : \ \Omega \to \mathbb{R}^m, \ \ \varphi = (\varphi_k)_{1\le k\le m}, \ \ \|\varphi\|_{L_p(\Omega)} = \left\| \max_{1\le k\le m} |\varphi_k| \right\|_{L_p(\Omega)}.$$

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Let f be a distribution on Ω,
$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$, $|\alpha| = \alpha_1 + \dots + \alpha_d$,

$$\nabla^r f = \left(\partial^r f / \partial x^\alpha\right)_{|\alpha|=r}, \quad m_r = \mathrm{card}\,\{\alpha : \ |\alpha| = r\}.$$

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We define

$$W^r_{p,g}(\Omega) = \Big\{ f : \ \Omega \to \mathbb{R} \Big| \ \exists \varphi : \ \Omega \to \mathbb{R}^{m_r} : \ \|\varphi\|_{L_p(\Omega)} \le 1, \ \nabla^r f = g \cdot \varphi \Big\}.$$

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Let

$$\|f\|_{L_{q,v}(\Omega)} = \|f\|_{q,v} = \|fv\|_{L_q(\Omega)}, \qquad L_{q,v}(\Omega) = \{f : \Omega \to \mathbb{R} | \ \ \|f\|_{q,v} < \infty\}.$$

Some classical results

Let Ω be a bounded domain with a Lipschitz boundary, $W^r_p(\Omega) = W^r_{p,1}(\Omega)$.

For $\omega > 0$, $1 \le p$, $q \le \infty$ we put $\eta_{p,q} = \frac{1}{2} \cdot \frac{1/p-1/q}{1/2-1/q}$,

- $\theta_{p,q,\omega} = \omega$, if $q \le p$ or $(2 \le p < q \le \infty,\ \omega > \eta_{p,q})$,
- $\theta_{p,q,\omega} = \omega + \frac{1}{q} - \frac{1}{p}$, if $1 \le p < q \le 2$,
- $\theta_{p,q,\omega} = \omega + \frac{1}{2} - \frac{1}{p}$, if $1 < p < 2 < q \le \infty$ and $p\omega > 1$,
- $\frac{q}{2}\left(\omega + \frac{1}{q} - \frac{1}{p}\right)$, if $(p < 2 < q,\ \omega < \frac{1}{p})$ or $(2 \le p < q,\ \omega < \eta_{p,q})$.

Theorem

(V.M. Tikhomirov, R.S. Ismagilov, Yu.I. Makovoz, V.E. Maiorov, B.S. Kashin, V.N. Temlyakov, E.M. Galeev, E.D. Kulanin). Let $1 < p \le \infty$, $1 \le q < \infty$, $\frac{r}{d} + \frac{1}{q} - \frac{1}{p} > 0$, and

- $\frac{r}{d} \ne \frac{1}{p}$, if $p < 2 < q$,
- $\frac{r}{d} \ne \eta_{p,q}$, if $2 \le p < q$.

Then

$$d_n(W^r_p(\Omega),\, L_q(\Omega)) \underset{r,d,p,q,\Omega}{\asymp} n^{-\theta_{p,q,r/d}}.$$

Let $L_{0,q,v}[a, b]$ be a space of measurable functions defined on $[a, b]$ with a seminorm $\|\cdot\|_{q,v}$.
We say that there exists a continuous embedding of $W^r_{p,g}[a, b]$ into $L_{q,v}[a, b]$ if there exists $n \in \mathbb{N}$ such that

$$d_n(W^r_{p,g}[a, b], L_{0,q,v}[a, b]) < \infty.$$

Let $L_{0,q,v}[a,\, b]$ be a space of measurable functions defined on $[a,\, b]$ with a seminorm $\|\cdot\|_{q,v}$.
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We define two-weighted Riemann – Liouville operators and their compositions:

$$\tilde{I}^{a,b,r}_{r,g,v} = (I_{r,g,v,a}f)(x) = v(x)\int\limits_a^x (x-t)^{r-1}g(t)f(t)\,dt,$$

$$\tilde{I}^{a,b,0}_{r,g,v} = (\tilde{I}_{r,g,v,b}f)(x) = v(x)\int\limits_x^b (t-x)^{r-1}g(t)f(t)\,dt,$$

$$\tilde{I}^{a,b,k}_{r,g,v} = I_{k,1,v,a}\circ \tilde{I}_{r-k,g,1,b}, \quad 1 \le k \le r-1.$$

- V.D. Stepanov, 1990: two-sided estimates of $\|I_{r,g,v,a}\|_{L_p[a,\,b]\to L_q[a,\,b]}$;
- H.P. Heinig, A. Kufner, 1990: two-sided estimates of $\|\tilde{I}^{a,b,k}_{r,g,v}\|_{L_p[a,\,b]\to L_q[a,\,b]}$, $1 \le k \le r-1$.

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Let $g \in L_1[a-\delta,\, b]$, $v \in L_1[a-\delta,\, b]$ for any $\delta \in (0,\, b-a)$.
A.V., 2009: there exists a continuous embedding of $W^r_{p,g}[a,\, b]$ into $L_{q,v}[a,\, b]$ if and only if there exists $k \in \{0,\, \dots,\, r\}$ such that

$$\|\tilde{I}^{a,b,k}_{r,g,v}\|_{L_p[a,\, b]\to L_q[a,\, b]} < \infty. \tag{1}$$

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$$\|\tilde{I}^{a,b,k}_{r,g,v}\|_{L_p[a,\,b]\to L_q[a,\,b]} < \infty. \qquad (1)$$

If (1) holds, then we put

$$\hat{W}^r_{p,g}[a,\, b] = \left\{\frac{1}{(k-1)!(r-1-k)!}\tilde{I}^{a,b,k}_{r,g,1}\varphi : \|\varphi\|_p \le 1\right\}.$$

Let $g, v : (0, e^{-1}] \to \mathbb{R}_+$,

$$g(x) = x^{-\beta_g} |\ln x|^{-\alpha_g} \rho_g(|\ln x|), \quad v(x) = x^{-\beta_v} |\ln x|^{-\alpha_v} \rho_v(|\ln x|),$$

where $\rho_g, \rho_v : [1, \infty) \to (0, +\infty)$ are absolutely continuous functions and

$$\lim_{y\to+\infty} \frac{y\rho_g'(y)}{\rho_g(y)} = \lim_{y\to\infty} \frac{y\rho_v'(y)}{\rho_v(y)} = 0.$$

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Let $\alpha = \alpha_g + \alpha_v,\ \rho(y) = \rho_g(y)\rho_v(y)$, Let $g, \ v : (0, \ e^{-1}] \to \mathbb{R}_+$,

$$g(x) = x^{-\beta_g}|\ln x|^{-\alpha_g}\rho_g(|\ln x|), \quad v(x) = x^{-\beta_v}|\ln x|^{-\alpha_v}\rho_v(|\ln x|),$$

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Let $\alpha = \alpha_g + \alpha_v, \ \rho(y) = \rho_g(y)\rho_v(y)$,

$$\alpha_{pq} = \begin{cases} \frac{1}{q} - \frac{1}{p}, \text{ if } q < p, \\ 0, \text{ if } p \le q \le 2, \\ \frac{1}{q}, \text{ if } p \le 2 < q, \\ \frac{1}{q}\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{2}-\frac{1}{q}}, \text{ if } 2 \le p \le q. \end{cases}$$

If

$$\beta_g + \beta_v = r + \frac{1}{q} - \frac{1}{p},\ \alpha > \max\left(\frac{1}{q} - \frac{1}{p}, 0\right) \\ \text{and } \beta_v \neq k + \frac{1}{q},\ \ k \in \{0, \dots, r-1\}, \tag{2}$$

then there exists a continuous embedding of $W^r_{p,g}[a,\ b]$ into $L_{q,v}[a,\ b]$.

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then there exists a continuous embedding of $W^r_{p,g}[a, b]$ into $L_{q,v}[a, b]$.

Theorem 1. Let (2) holds, $p > 1$, $q < \infty$.

- If $\alpha > r + \frac{1}{q} - \frac{1}{p}$ then

$$d_n(\hat{W}^r_{p,g}[0, e^{-1}], L_{q,v}[0, e^{-1}]) \asymp n^{-\theta_{p,q,r}}.$$

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- If $\alpha_{pq} < \alpha < r + \frac{1}{q} - \frac{1}{p}$ then

$$d_n(\hat{W}^r_{p,g}[0, e^{-1}], L_{q,v}[0, e^{-1}]) \asymp \begin{cases} \rho(n) n^{-\alpha + \frac{1}{q} - \frac{1}{p}}, \text{ if } p \geq q, \\ \rho(n) n^{-\alpha - \alpha_{pq}\left(\frac{q}{2} - 1\right)}, \text{ if } p < q. \end{cases}$$

- If $q > 2$, $p < q$ and $0 < \alpha < \alpha_{pq}$, then

$$d_n(\hat{W}^r_{p,g}[0,\, e^{-1}],\, L_{q,v}[0,\, e^{-1}]) \asymp \rho(n^{q/2}) n^{-\frac{\alpha q}{2}}.$$

- If $q > 2$, $p < q$ and $0 < \alpha < \alpha_{pq}$, then

$$d_n(\hat{W}^r_{p,g}[0,\ e^{-1}],\ L_{q,v}[0,\ e^{-1}]) \asymp \rho(n^{q/2})n^{-\frac{\alpha q}{2}}.$$

- If $\rho_g = \rho_v = 1$ and $\alpha = r + \frac{1}{q} - \frac{1}{p}$ then

$$d_n(\hat{W}^r_{p,g}[0,\ e^{-1}],\ L_{q,v}[0,\ e^{-1}]) \asymp (\ln n)^{r+\frac{1}{q}-\frac{1}{p}} n^{-\theta_{p,q,r}}.$$

Let $w : \mathbb{R}^d \to (0, +\infty)$, $0 < p < +\infty$. For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we put

$$\|f\|_{L_p(w)} = \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)\, dx \right)^{1/p},$$

$\|f\|_{L_\infty(w)} = \|f\|_{L_\infty}$.

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Notations:

- $\mathcal{S}(\mathbb{R}^d)$ — the Schwartz space,
- $\mathcal{S}'(\mathbb{R}^d)$ — its dual,
- $f \mapsto \mathcal{F}(f)$ — the Fourier transform,
- $\mathcal{A}_\infty$ — class of Muckenhoupt weights.

Let $\varphi \in S(\mathbb{R}^d)$,

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R}^d : |y| < 2\}, \quad \varphi(x) = 1 \text{ if } |x| \le 1,$$

$\varphi_0 = \varphi$, $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$.
Let $0 < p \le \infty$, $0 < q \le \infty$, $w \in \mathcal{A}_\infty$. The weighted Besov space $B^s_{p,q}(\mathbb{R}^d, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d, w)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^q_{L_p(w)} \right)^{1/q} < \infty.$$

Notations:

- $Q_{\nu,\mathbf{m}}$ — the d-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$;
- $w(Q_{\nu,\mathbf{m}}) = \int\limits_{Q_{\nu,\mathbf{m}}} w(x)\,dx$.

Let $\sigma \in \mathbb{R}$, $w \in \mathcal{A}_\infty$, $\lambda_{\nu,\mathbf{m}} \in \mathbb{C}$, $\nu \in \mathbb{Z}_+$, $\mathbf{m} \in \mathbb{Z}^d$,

$$\chi_\nu^{(p)}(x) = \begin{cases} 2^{\frac{\nu d}{p}}, & x \in Q_{\nu,\mathbf{m}}, \\ 0, & x \notin Q_{\nu,\mathbf{m}}, \end{cases}$$

$$\|(\lambda_{\nu,\mathbf{m}})_{\nu\in\mathbb{Z}_+,\,\mathbf{m}\in\mathbb{Z}^d}\|_{b^\sigma_{p,q}(w)} = \left(\sum_{\nu=0}^{\infty} 2^{\nu\sigma q}\left\|\sum_{\mathbf{m}\in\mathbb{Z}^d} \lambda_{\nu,\mathbf{m}}\chi^{(p)}_{\nu,\mathbf{m}}\right\|^q_{L_p(w)}\right)^{1/q},$$

$$b^\sigma_{p,q}(w) = \left\{(\lambda_{\nu,\mathbf{m}})_{\nu\in\mathbb{Z}_+,\,\mathbf{m}\in\mathbb{Z}^d} :\ \|(\lambda_{\nu,\mathbf{m}})_{\nu\in\mathbb{Z}_+,\,\mathbf{m}\in\mathbb{Z}^d}\|_{b^\sigma_{p,q}(w)} < \infty\right\},$$

$$\sigma(s,\ p) := s + \tfrac{d}{2} - \tfrac{d}{p}.$$

Theorem

(D.D. Haroske, L. Skrzypczak, 2008). Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, p, $q \in (0, +\infty)$. Then there exists an isomorphism $T_{s,p,q,w} : B^s_{p,q}(\mathbb{R}^d, w) \to b^{\sigma(s,\,p)}_{p,q}(w)$. For any w_1, $w_2 \in \mathcal{A}_\infty$, $s_i \in \mathbb{R}$, p_i, $q_i \in (0, +\infty)$, $i = 1, 2$, such that $\mathrm{Id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^d, w_2)$ or $\mathrm{id} : b^{\sigma(s_1,\,p_1)}_{p_1,q_1}(w_1) \to b^{\sigma(s_2,\,p_2)}_{p_2,q_2}(w_2)$ is bounded, the diagram

$$\begin{array}{ccc} B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1) & \xrightarrow{T_{s_1,p_1,q_1,w_1}} & b^{\sigma(s_1,\,p_1)}_{p_1,q_1}(w_1) \\ \mathrm{Id}\Big\downarrow & & \mathrm{id}\Big\downarrow \\ B^{s_2}_{p_2,q_2}(\mathbb{R}^d, w_2) & \xrightarrow{T_{s_2,p_2,q_2,w_2}} & b^{\sigma(s_2,\,p_2)}_{p_2,q_2}(w_2) \end{array}$$

is commutative.

Theorem

(Th. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak, D.D. Haroske) Let $s_i \in \mathbb{R}$, p_i, $q_i \in (0, +\infty)$, $i = 1, 2$. Then

$$\|\mathrm{Id}\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d,\, w_1)\to B^{s_2}_{p_2,q_2}(\mathbb{R}^d,\, w_2)} \asymp \|\mathrm{id}\|_{b^{\sigma(s_1,\, p_1)}_{p_1,q_1}(w_1)\to b^{\sigma(s_2,\, p_2)}_{p_2,q_2}(w_2)} \asymp$$

$$\asymp \left\| \left\{ \left\| \left\{ 2^{-\nu(s_1-s_2)}(w_1(Q_{\nu,\mathbf{m}}))^{-1/p_1}(w_2(Q_{\nu,\mathbf{m}}))^{1/p_2} \right\}_{\mathbf{m}\in\mathcal{M}_\nu} \right\|_{p^*} \right\}_{\nu\in\mathbb{Z}_+} \right\|_{l_{q^*}},$$

where

$$\frac{1}{p_*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+, \quad \frac{1}{q_*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+.$$

Let

- $s_1, s_2 \in \mathbb{R}$, $1 < p_1, q_1 \le +\infty$, $1 \le p_2, q_2 < +\infty$, $\delta := s_1 - s_2 + \frac{d}{p_2} - \frac{d}{p_1} > 0$,

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- $\beta_g > -\frac{d}{p_1}$, $\beta_v < \frac{d}{p_2}$, $\beta_g + \beta_v = \delta$,

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- $\beta_g > -\frac{d}{p_1}$, $\beta_v < \frac{d}{p_2}$, $\beta_g + \beta_v = \delta$,
- $\gamma_g > -\frac{d}{p_1}$, $\gamma_v < \frac{d}{p_2}$, $\gamma := \gamma_g + \gamma_v > \delta$,

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- $\beta_g > -\frac{d}{p_1},\ \beta_v < \frac{d}{p_2},\ \beta_g + \beta_v = \delta,$
- $\gamma_g > -\frac{d}{p_1},\ \gamma_v < \frac{d}{p_2},\ \gamma := \gamma_g + \gamma_v > \delta,$
- $\alpha_g > 0,\ \alpha_v > 0,\ \alpha = \alpha_g + \alpha_v,$

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- $\gamma_g > -\frac{d}{p_1}$, $\gamma_v < \frac{d}{p_2}$, $\gamma := \gamma_g + \gamma_v > \delta$,
- $\alpha_g > 0$, $\alpha_v > 0$, $\alpha = \alpha_g + \alpha_v$,
- $\rho_g, \rho_v : [1, +\infty) \to (0, +\infty)$ be absolutely continuous functions such that $\lim_{y\to+\infty} \frac{y\rho_g'(y)}{\rho_g(y)} = \lim_{y\to+\infty} \frac{y\rho_v'(y)}{\rho_v(y)} = 0$,

Let

- $s_1, s_2 \in \mathbb{R}$, $1 < p_1, q_1 \le +\infty$, $1 \le p_2, q_2 < +\infty$, $\delta := s_1 - s_2 + \frac{d}{p_2} - \frac{d}{p_1} > 0$,
- $\beta_g > -\frac{d}{p_1}$, $\beta_v < \frac{d}{p_2}$, $\beta_g + \beta_v = \delta$,
- $\gamma_g > -\frac{d}{p_1}$, $\gamma_v < \frac{d}{p_2}$, $\gamma := \gamma_g + \gamma_v > \delta$,
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- $\rho(y) = \rho_g(y)\rho_v(y)$,
- $g(x) = |x|^{-\beta_g}|\log_2|x||^{-\alpha_g}\rho_g(|\log_2|x||)$, $v(x) = |x|^{-\beta_v}|\log_2|x||^{-\alpha_v}\rho_v(|\log_2|x||)$, if $|x| \le \frac{1}{2}$,

Let

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- $g(x) = |x|^{-\gamma_g},\ v(x) = |x|^{-\gamma_v}$, if $|x| > \frac{1}{2},$
- $w_1(x) = g^{-p_1}(x),\ w_2(x) = v^{p_2}(x).$

Theorem 2.

Let $1 < p_1 \le p_2 < \infty$, $1 < q_1 \le q_2 < \infty$, $p_2 \ge 2$, $q_2 \ge 2$,

$$\lambda(\mathbf{p}) = \min\left\{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, 1\right\}, \quad \lambda(\mathbf{q}) = \min\left\{\frac{\frac{1}{q_1}-\frac{1}{q_2}}{\frac{1}{2}-\frac{1}{q_2}}, 1\right\},$$

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$\theta_1 = \frac{\delta}{d} + \frac{\lambda(\mathbf{p})}{2} - \frac{\lambda(\mathbf{p})}{p_2}$, $\theta_2 = \frac{p_2\delta}{2d}$, $\theta_3 = \alpha + \frac{\lambda(\mathbf{q})}{2} - \frac{\lambda(\mathbf{q})}{q_2}$, $\theta_4 = \frac{\alpha q_2}{2}$,
$\sigma_1 = \sigma_2 = 0$, $\sigma_3 = 1$, $\sigma_4 = \frac{q_2}{2}$.

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Let $1 \le j_* \le 4$, $\theta_{j_*} < \min\{\theta_j,\ 1 \le j \le 4,\ j \ne j_*\}$.

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Let $1 \le j_* \le 4$, $\theta_{j_*} < \min\{\theta_j,\ 1 \le j \le 4,\ j \ne j_*\}$. Then

$$d_n(B^{s_1}_{p_1,q_1}(\mathbb{R}^d,\, w_1),\, B^{s_2}_{p_2,q_2}(\mathbb{R}^d,\, w_2)) \asymp n^{-\theta_{j_*}}\rho(n^{\sigma_{j_*}}).$$

Let m, $k \in \mathbb{N}$, $1 \le p$, $q < \infty$. Let $l_{p,q}^{m,k}$ be the space $\mathbb{R}^{mk}$ with a norm

$$\|(x_{ij})_{1\le i\le m,\, 1\le j\le k}\|_{l_{p,q}^{m,k}} = \left(\sum_{j=1}^{k}\left(\sum_{i=1}^{m}|x_{ij}|^p\right)^{q/p}\right)^{1/q},$$

$$B_{p,q}^{m,k} = \{x \in l_{p,q}^{m,k} : \|x\|_{l_{p,q}^{m,k}} \le 1\}.$$

Theorem 3.

Let $1 \le p_1 \le p_2 < \infty$, $1 \le q_1 \le q_2 < \infty$, $p_2 \ge 2$, $q_2 \ge 2$. Then there exists $a = a(p_1, q_1, p_2, q_2) > 0$ such that for any m, k, $n \in \mathbb{N}$, $n \le amk$, we have

$$d_n(B^{m,k}_{p_1,q_1}, l^{m,k}_{p_2,q_2}) \underset{p_1,p_2,q_1,q_2}{\asymp} \Phi_0 = \Phi_0(m, k, n, p_1, p_2, q_1, q_2),$$

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if $\lambda(\mathbf{p}) \ge \lambda(\mathbf{q})$ and $\lambda(\mathbf{q}) < 1$.

Thank you for the attention!

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