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A low-frequency asymptotic expansion of a strong solution to an initial-boundary value problem

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Abstract. We study a low-frequency asymptotic expansion for a *strong* solution to an initial-boundary value problem of a semi-linear wave equation. This equation admits space-time dependent coefficients and a memory boundary-like antiperiodic condition. For some small parameters from coefficients of this semi-linear wave equation and of boundary conditions, we approximate a unique strong solution to this problem by a polynomial of these parameters; and coefficients of this polynomial are strong solutions of a sequence of well-defined linear initial-boundary value problems.

1. Introduction

For $\Omega = (0,1)$ and $\Omega_T = \Omega \times (0,T)$, $T > 0$, we consider the problem

$$\frac{\partial}{\partial t}\left(a(x,t)\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right) + KG\left(u\right) + \lambda H\left(\frac{\partial u}{\partial t}\right) = f(x,t) \text{ in } \Omega_T, \tag{1.1}$$

$$b(0,t)\frac{\partial u}{\partial x}(0,t) = B_0(t) \text{ in } (0,T), \tag{1.2}$$

$$-b(1,t)\frac{\partial u}{\partial x}(1,t) = B_1(t) \text{ in } (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega, \tag{1.4}$$

where $u = u(x,t)$, $(x,t) \in \Omega_T$, is unknown; a, b, G, H, f, u_0 and u_1 are given functions. The boundary values B_0 and B_1 are defined by

$$\begin{aligned} B_0(t) = & a_0 u(0,t) + b_0 \frac{\partial u}{\partial t}(0,t) + c_1 u(1,t) + d_1 \frac{\partial u}{\partial t}(1,t) + \\ & + p_0(t) + \int_0^t q_0(t-s) u(0,s) ds + \int_0^t r_1(t-s) u(1,s) ds, \end{aligned} \tag{1.5}$$

$$\begin{aligned} B_1(t) = & a_1 u(1,t) + b_1 \frac{\partial u}{\partial t}(1,t) + c_0 u(0,t) + d_0 \frac{\partial u}{\partial t}(0,t) + \\ & + p_1(t) + \int_0^t q_1(t-s) u(1,s) ds + \int_0^t r_0(t-s) u(0,s) ds, \end{aligned} \tag{1.6}$$

here a_0, a_1, b_0, b_1, c_0, c_1, d_0 and d_1 are given constants; p_0, p_1, q_0, q_1, r_0 and r_1 are given functions.

In (1.2), (1.3), (1.5) and (1.6), if

$$a_0 = c_0,\ a_1 = c_1,\ b_0 = d_0,\ b_1 = d_1,$$

$$p_0(t) = p_1(t),\ q_0(t) = q_1(t),\ r_0(t) = r_1(t),$$

then it follows that

$$b(0,t)\frac{\partial u}{\partial x}(0,t) = -b(1,t)\frac{\partial u}{\partial x}(1,t) \text{ in } (0,T).$$

Hence, the boundary values given by (1.2), (1.3), (1.5) and (1.6) define the so-called *memory boundary-like antiperiodic condition.*

In [Lê, U.V.: A semi-linear wave equation with space-time dependent coefficients and a memory boundary-like antiperiodic condition: regularity and stability. *J. Math. Phys.* **51** (2010) 103504, 28 pp], we proved that problem $\{(1.1)\text{-}(1.6)\}$ has a unique strong solution.

Now, for some small parameters from coefficients of the above semi-linear wave equation and of the boundary conditions, we approximate this unique strong solution to this problem by a polynomial of these parameters; and coefficients of this polynomial are strong solutions of a sequence of well-defined linear initial-boundary value problems.

2. Preliminaries

We omit definitions of usual function spaces C^m, L^p, $W^{m,p}$, H^m for $p \in [1, +\infty]$ and $m \in \mathbb{N}$. Here are some other notations used in this paper.

- $\Omega = (0, 1)$, $\Omega_T \equiv \Omega \times (0, T)$ and $\mathbb{R}_* = \mathbb{R}_+ \cup \{0\}$.
- $\langle \cdot, \cdot \rangle$: the scalar product in $L^2(\Omega)$.
- $\| \cdot \|_X$: the norm of a Banach space X.

- ▷ $L^p(0,T;X)$, $1 \le p \le +\infty$, $T > 0$: the Banach space of the real measurable functions $w : (0,T) \to X$ such that

$$\|w\|_{L^p(0,T;X)} = \left(\int_0^T \|w(\cdot,t)\|_X^p dt \right)^{1/p} < +\infty \quad \text{for} \quad 1 \le p < +\infty,$$

and

$$\|w\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|w(\cdot,t)\|_X < +\infty \quad \text{for} \quad p = +\infty.$$

- $W^{m,p}(0,T;X)$, $p \in [1,+\infty]$, $m \in \mathbb{N}$: the Sobolev spaces of all functions $w \in L^p(0,T;X)$ such that $w^{(m)}$ exist in a weak sense, and belong to $L^p(0,T;X)$; and

$$\|w\|_{W^{m,p}(0,T;X)} := \begin{cases} \left(\int_0^T \sum_{i=0}^m \|w^{(i)}(\cdot,t)\|_X^p dt \right)^{1/p} & 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}_{0 \leq t \leq T} \sum_{i=0}^m \|w^{(i)}(\cdot,t)\|_X & p = +\infty. \end{cases}$$

- $H^m(0,T;X) \equiv W^{m,2}(0,T;X)$.

- For any $T > 0$, we consider Banach spaces

$$\mathbb{H}^1(\Omega_T) = \left\{ y \in H^1\left(0, T; L^2(\Omega)\right) \cap L^\infty\left(0, T; H^1(\Omega)\right) \,\middle|\, \frac{\partial y}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right) \right\},$$

$$\mathbb{H}^2(\Omega_T) = \left\{ y \in H^2\left(0, T; L^2(\Omega)\right) \cap H^1\left(\Omega_T\right) \,\middle|\, \frac{\partial y}{\partial t} \in L^\infty\left(0, T; H^1(\Omega)\right), \frac{\partial^2 y}{\partial t^2} \in L^\infty\left(0, T; L^2(\Omega)\right) \right\}$$

endowed with the norms

$$\|y\|^2_{\mathbb{H}^1(\Omega_T)} = \|y\|^2_{H^1(0,T;L^2(\Omega))} + \|y\|^2_{L^\infty(0,T;H^1(\Omega))} + \left\|\frac{\partial y}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))},$$

$$\begin{aligned}\|y\|^2_{\mathbb{H}^2(\Omega_T)} =& \|y\|^2_{H^2(0,T;L^2(\Omega))} + \|y\|^2_{H^1(\Omega_T)} + \\ &+ \left\|\frac{\partial y}{\partial t}\right\|^2_{L^\infty(0,T;H^1(\Omega))} + \left\|\frac{\partial^2 y}{\partial t^2}\right\|^2_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

respectively, for $y \in \mathbb{H}^1(\Omega_T) \cup \mathbb{H}^2(\Omega_T)$.

Definition 1. A function $u \in \mathbb{H}^1(\Omega_T)$ is a weak solution to problem $\{$(1.1)-(1.6)$\}$ if $u(x,t)$ satisfies the following variational problem:

$$\frac{d}{dt}\left\langle a(\cdot,t)\frac{\partial u}{\partial t}(\cdot,t),\varphi\right\rangle + \left\langle b(\cdot,t)\frac{\partial u}{\partial x}(\cdot,t),\varphi'\right\rangle + B_0(t)\varphi(0) + B_1(t)\varphi(1) +$$

$$+\left\langle G\left(u(\cdot,t)\right),\varphi\right\rangle + \left\langle H\left(\frac{\partial u}{\partial t}(\cdot,t)\right),\varphi\right\rangle = \left\langle f(\cdot,t),\varphi\right\rangle, \tag{2.1}$$

$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{2.2}$$

$$b(0,t)\frac{\partial u}{\partial x}(0,t) = B_0(t),\ -b(1,t)\frac{\partial u}{\partial x}(1,t) = B_1(t), \tag{2.3}$$

$$B_0(t) = a_0 u(0,t) + b_0 \frac{\partial u}{\partial t}(0,t) + c_1 u(1,t) + d_1 \frac{\partial u}{\partial t}(1,t) + p_0(t) +$$

$$+ \int_0^t q_0(t-s)u(0,s)ds + \int_0^t r_1(t-s)u(1,s)ds, \tag{2.4}$$

$$B_1(t) = a_1 u(1,t) + b_1 \frac{\partial u}{\partial t}(1,t) + c_0 u(0,t) + d_0 \frac{\partial u}{\partial t}(0,t) + p_1(t) +$$

$$+ \int_0^t q_1(t-s)u(1,s)ds + \int_0^t r_0(t-s)u(0,s)ds, \tag{2.5}$$

for each $\varphi \in H^1(\Omega)$, a.e. time $0 \leq t \leq T$; where $\frac{d}{dt}\left\langle a(\cdot,t)\frac{\partial u}{\partial t}(\cdot,t), \varphi\right\rangle$ is the derivative in sense of distribution on $(-\infty, T)$ of

$$\begin{cases} \left\langle a(\cdot,t)\frac{\partial u}{\partial t}(\cdot,t), \varphi\right\rangle, \ t > 0, \\ 0, \ t < 0. \end{cases}$$

Definition 2. A weak solution u of problem $\{(1.1)\text{-}(1.6)\}$ is called a strong solution to this problem if $u \in \mathbb{H}^2(\Omega_T)$.

Definition 3. A trio $(a, b, c) \in \mathbb{R}^3$ is said to satisfy the EP-condition if the pair $(a, c) \in \mathbb{R}_+^2$ and "the elliptic condition" $b^2 - 4ac < 0$ holds.

Lemma 1. *Let a trio $(a, b, c) \in \mathbb{R}^3$ fulfil the EP-condition. Then there exists a positive constant*

$$d \leq \frac{a + c - \sqrt{(a-c)^2 + b^2}}{2}$$

such that the inequality

$$ax^2 + bxy + cy^2 \geq d(x^2 + y^2)$$

holds for all $(x, y) \in \mathbb{R}^2$.

Now we introduce an important equality for studying the low-frequency asymptotic expansion. For $N \in \mathbb{Z}_+$, define a partial order "$\leq$" on $\mathbb{Z}^N$: with $\overrightarrow{i} = (i_1, i_2, \cdots, i_N)$, $\overrightarrow{j} = (j_1, j_2, \cdots, j_N) \in \mathbb{Z}^N$,

$$\overrightarrow{i} \leq \overrightarrow{j} \Leftrightarrow \begin{cases} i_1 \leq j_1, \\ i_2 \leq j_2, \\ \cdots\cdots, \\ i_N \leq j_N. \end{cases}$$

In addition, here are some definitions on $\mathbb{Z}_*^N$ with $\mathbb{Z}_* = \mathbb{Z}_+ \cup \{0\}$ and $\mathbb{Z}_+ = \{z \in \mathbb{Z} : z > 0\}$:

- ▷ $\overrightarrow{i}\,! := i_1! i_2! \cdots i_N!$,
- ▷ $|\overrightarrow{i}\,| := i_1 + i_2 + \cdots + i_N$,
- ▷ $C_{\overrightarrow{i}}^{\overrightarrow{j}} := \frac{\overrightarrow{i}\,!}{\overrightarrow{j}\,!(\overrightarrow{i} - \overrightarrow{j}\,)!}$.

With $\overrightarrow{\eta} = (\eta_1, \eta_2, \cdots, \eta_N) \in \mathbb{R}^N$, denote

- ▷ $\overrightarrow{\eta}^{\overrightarrow{i}} := \eta_1^{i_1} \eta_2^{i_2} \cdots \eta_N^{i_N}$.

Lemma 2. *Let* $(m, n, N, \overrightarrow{i}) \in \mathbb{Z}_+^3 \times \mathbb{Z}_+^N$. *The equality*

$$\left(\sum_{1 \leq |\overrightarrow{i}| \leq n} a_{\overrightarrow{i}} \overrightarrow{\eta}^{\overrightarrow{i}} \right)^m = \sum_{m \leq |\overrightarrow{i}| \leq mn} \left[a_{\overrightarrow{i}} \right]_m \overrightarrow{\eta}^{\overrightarrow{i}}$$

holds for $(a_{\overrightarrow{i}}, \overrightarrow{\eta}) \in \mathbb{R} \times \mathbb{R}^N$, *where*

$$\left[a_{\overrightarrow{i}} \right]_m := \begin{cases} a_{\overrightarrow{i}},\ |\overrightarrow{i}| \leq n,\ m = 1, \\ \displaystyle\sum_{\overrightarrow{j} \in \left[\mathbb{Z}_{\overrightarrow{i}} \right]_m} a_{\overrightarrow{i} - \overrightarrow{j}} \left[a_{\overrightarrow{j}} \right]_{m-1},\ m \leq |\overrightarrow{i}| \leq mn,\ m \geq 2, \end{cases}$$

with

$$\left[\mathbb{Z}_{\overrightarrow{i}} \right]_m := \big\{ \overrightarrow{j} \in \mathbb{Z}_+^N : \overrightarrow{j} \leq \overrightarrow{i}, 1 \leq |\overrightarrow{i}| - |\overrightarrow{j}| \leq n, \\ m - 1 \leq |\overrightarrow{j}| \leq (m-1)n \big\}.$$

3. Assumptions for the given data

3.1. Main assumptions

$\overleftarrow{\triangleright}$ $(\overleftarrow{A}_G)$: This includes two conditions for the internal source term G.

1^0. $|G(v)| \leq |v|^\alpha$ for $(v, \alpha) \in \mathbb{R} \times \mathbb{R}_+$;

2^0. $G \in C^1([-M, M])$ for $M \in \mathbb{R}_+$.

$\overleftarrow{\triangleright}$ $(\overleftarrow{A}_H)$: The internal damping $H : \mathbb{R} \to \mathbb{R}$ satisfies

1^0. $H(v)v \geq 0$ for $v \in \mathbb{R}$;

2^0. $|H(v)| \leq C(|v| + 1)$ for $(v, C) \in \mathbb{R} \times \mathbb{R}_*$, and $H \in C^1(\mathbb{R})$ and $|H'(w)| \leq |w|^\beta$ for $(w, \beta) \in \mathbb{R} \times \mathbb{R}_+$.

- $\overleftarrow{\triangleright}$ $(\overleftarrow{A}_{b_*d_*})$: The trio $(b_0, d_0 + d_1, b_1)$ satisfies the EP-condition.

- $\overset{\hookrightarrow}{\triangleright}$ $(\overset{\hookrightarrow}{A}_u)$: The pair (u_0, u_1) belongs to $H^2(\Omega) \times H^1(\Omega)$, and fulfills the compatibility condition

$$\begin{cases} b(0,0)u_0'(0) = a_0u_0(0) + b_0u_1(0) + c_1u_0(1) + d_1u_1(1) + p_0(0), \\ -b(1,0)u_0'(1) = a_1u_0(1) + b_1u_1(1) + c_0u_0(0) + d_0u_1(0) + p_1(0), \end{cases}$$

where the quartet (b_0, b_1, d_0, d_1) satisfies the assumption $(\overleftarrow{A}_{b_*d_*})$. $\triangleleft$

3.2. Technical conditions

$\overset{\hookrightarrow}{\triangleright}$ $(\overset{\hookrightarrow}{A}_{ab})$:

$$\left(a, b, \frac{\partial^2 a}{\partial t^2}, \frac{\partial^2 b}{\partial t^2}\right) \in \left(C^1\left([0,T]; L^\infty(\Omega)\right)\right)^2 \times \left(L^1\left(0,T; L^\infty(\Omega)\right)\right)^2,$$

$\frac{\partial b}{\partial x}(\cdot, t) \in L^\infty(\Omega)$ and $a(x,t) \geq \overline{a} > 0,\ b(x,t) \geq \overline{b} > 0$ for $(x,t) \in \Omega_T$.

$\overset{\hookrightarrow}{\triangleright}$ $(\overset{\hookrightarrow}{A}_f)$: $\left(f, \frac{\partial f}{\partial t}\right) \in L^2\left(\Omega_T\right) \times L^2\left(\Omega_T\right)$.

$\triangleright$ $(A_{K\lambda})$: $(K, \lambda) \in \mathbb{R} \times \mathbb{R}_+$.

$\triangleright$ $(A_{a_* c_*})$: $(a_0, a_1, c_0, c_1) \in \mathbb{R}^4$.

$\overset{\hookrightarrow}{\triangleright}$ $(\overset{\hookrightarrow}{A}_{pqr})$: $(p_0, p_1, q_0, q_1, r_0, r_1) \in \left(H^1(0,T)\right)^6$. $\triangleleft$

For $N \in \mathbb{N}$ we define a family of solvable initial-boundary value problems $\left(\mathbb{P}_{\overrightarrow{i}}\right)$, where $\overrightarrow{i} \in \mathbb{Z}_*^6$ and $\left|\overrightarrow{i}\right| \leq N$. For this purpose, we consider the following two cases of $\overrightarrow{i}$.

Case 1: $\left|\overrightarrow{i}\right| = 0$ or $\overrightarrow{i} = \overrightarrow{0}$.

We have the problem $\left(\mathbb{P}_{\overrightarrow{0}}\right)$ given by

$$\left(\mathbb{P}_{\overrightarrow{0}}\right) \begin{cases} \dfrac{\partial}{\partial t}\left(a(x,t)\dfrac{\partial u_{\overrightarrow{0}}}{\partial t}\right) - \dfrac{\partial}{\partial x}\left(b(x,t)\dfrac{\partial u_{\overrightarrow{0}}}{\partial x}\right) = f(x,t) \equiv \square_{\overrightarrow{0}}, \\ b(0,t)\dfrac{\partial u_{\overrightarrow{0}}}{\partial x}(0,t) = B_{\overrightarrow{0}}^{(0)}(t), \\ -b(1,t)\dfrac{\partial u_{\overrightarrow{0}}}{\partial x}(1,t) = B_{\overrightarrow{0}}^{(1)}(t), \\ u_{\overrightarrow{0}}(x,0) = u_0(x),\ \dfrac{\partial u_{\overrightarrow{0}}}{\partial t}(x,0) = u_1(x), \end{cases} \tag{3.1}$$

with

$$B^{(0)}_{\overrightarrow{0}}(t) = b_0 \frac{\partial u_{\overrightarrow{0}}}{\partial t}(0,t) + d_1 \frac{\partial u_{\overrightarrow{0}}}{\partial t}(1,t) + p_0(t) + \int_0^t q_0(t-s) u_{\overrightarrow{0}}(0,s) ds + \int_0^t r_1(t-s) u_{\overrightarrow{0}}(1,s) ds, \tag{3.2}$$

$$B^{(1)}_{\overrightarrow{0}}(t) = b_1 \frac{\partial u_{\overrightarrow{0}}}{\partial t}(1,t) + d_0 \frac{\partial u_{\overrightarrow{0}}}{\partial t}(0,t) + p_1(t) + \int_0^t q_1(t-s) u_{\overrightarrow{0}}(1,s) ds + \int_0^t r_0(t-s) u_{\overrightarrow{0}}(0,s) ds. \tag{3.3}$$

Case 2: $1 \leq |\overrightarrow{i}| \leq N$.

The problems $\left(\mathbb{P}_{\overrightarrow{i}}\right)$ are defined by

$$\left(\mathbb{P}_{\overrightarrow{i}}\right)\begin{cases}\frac{\partial}{\partial t}\left(a(x,t)\frac{\partial u_{\overrightarrow{i}}}{\partial t}\right)-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial u_{\overrightarrow{i}}}{\partial x}\right)=\Box_{\overrightarrow{i}},\\ b(0,t)\frac{\partial u_{\overrightarrow{i}}}{\partial x}(0,t)=B^{(0)}_{\overrightarrow{i}}(t),\\ -b(1,t)\frac{\partial u_{\overrightarrow{i}}}{\partial x}(1,t)=B^{(1)}_{\overrightarrow{i}}(t),\\ u_{\overrightarrow{i}}(x,0)=u_0(x),\ \frac{\partial u_{\overrightarrow{i}}}{\partial t}(x,0)=u_1(x),\end{cases} \tag{3.4}$$

with

$$
\begin{aligned}
B^{(0)}_{\overrightarrow{i}}(t) =& \diamondsuit^{(0)}_{\overrightarrow{i}}(t) + b_0 \frac{\partial u_{\overrightarrow{i}}}{\partial t}(0,t) + d_1 \frac{\partial u_{\overrightarrow{i}}}{\partial t}(1,t) \\
&+ \int_0^t q_0(t-s) u_{\overrightarrow{i}}(0,s) ds + \int_0^t r_1(t-s) u_{\overrightarrow{i}}(1,s) ds,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
B^{(1)}_{\overrightarrow{0}}(t) =& \diamondsuit^{(1)}_{\overrightarrow{i}}(t) + b_1 \frac{\partial u_{\overrightarrow{i}}}{\partial t}(1,t) + d_0 \frac{\partial u_{\overrightarrow{i}}}{\partial t}(0,t) \\
&+ \int_0^t q_1(t-s) u_{\overrightarrow{i}}(1,s) ds + \int_0^t r_0(t-s) u_{\overrightarrow{i}}(0,s) ds.
\end{aligned} \tag{3.6}
$$

The functions $\square_{\overrightarrow{i}}$ in (3.4), and $\diamondsuit^{(0)}_{\overrightarrow{i}}$ and $\diamondsuit^{(1)}_{\overrightarrow{i}}$ in (3.5)–(3.6) are formed as follows:

$$\Box_{\overrightarrow{i}} = \begin{cases} 0, \quad \overrightarrow{i} \in \{(0,0)\} \times \mathbb{Z}_*^4, \\ -G\left(u_{\overrightarrow{0}}\right), \quad \overrightarrow{i} = \overrightarrow{e}_1, \\ -H\left(\frac{\partial u_{\overrightarrow{0}}}{\partial t}\right), \quad \overrightarrow{i} = \overrightarrow{e}_2, \\ -\sum_{r=1}^{\left|\overrightarrow{i}\right|-1} \frac{1}{r!} G^{(r)}\left(u_{\overrightarrow{0}}\right)\left[u_{\overrightarrow{i}-\overrightarrow{e}_1}\right]_r, \, 2 \le |\overrightarrow{i}|, \\ \qquad \overrightarrow{i} \in \{m \in \mathbb{Z}_+ : m \ge 1\} \times \{0\} \times \mathbb{Z}_*^4, \\ -\sum_{r=1}^{\left|\overrightarrow{i}\right|-1} \frac{1}{r!} H^{(r)}\left(\frac{\partial u_{\overrightarrow{0}}}{\partial t}\right)\left[\frac{\partial u_{\overrightarrow{i}-\overrightarrow{e}_2}}{\partial t}\right]_r, \, 2 \le |\overrightarrow{i}|, \\ \qquad \overrightarrow{i} \in \{0\} \times \{m \in \mathbb{Z}_+ : m \ge 1\} \times \mathbb{Z}_*^4, \\ -\sum_{r=1}^{\left|\overrightarrow{i}\right|-1} \frac{1}{r!} \left(G^{(r)}\left(u_{\overrightarrow{0}}\right)\left[u_{\overrightarrow{i}-\overrightarrow{e}_1}\right]_r + H^{(r)}\left(\frac{\partial u_{\overrightarrow{0}}}{\partial t}\right)\left[\frac{\partial u_{\overrightarrow{i}-\overrightarrow{e}_2}}{\partial t}\right]_r\right), \\ \qquad \overrightarrow{i} \in \{m \in \mathbb{Z}_+ : m \ge 1\}^2 \times \mathbb{Z}_*^4, \end{cases} \tag{3.7}$$

$$\Diamond^{(0)}_{\overrightarrow{i}}(t) = \begin{cases} 0, & \overrightarrow{i} \in \mathbb{Z}^2_* \times \{(0,0)\} \times \mathbb{Z}^2_*, \\ u_{\overrightarrow{i}-\overrightarrow{e}_3}(0,t), & \overrightarrow{i} \in \mathbb{Z}^2_* \times \{m \in \mathbb{Z}_+ : m \geq 1\} \times \{0\} \times \mathbb{Z}^2_*, \\ u_{\overrightarrow{i}-\overrightarrow{e}_4}(1,t), & \overrightarrow{i} \in \mathbb{Z}^2_* \times \{0\} \times \{m \in \mathbb{Z}_+ : m \geq 1\} \times \mathbb{Z}^2_*, \\ u_{\overrightarrow{i}-\overrightarrow{e}_3}(0,t) + u_{\overrightarrow{i}-\overrightarrow{e}_4}(1,t), & \overrightarrow{i} \in \mathbb{Z}^2_* \times \{m \in \mathbb{Z}_+ : m \geq 1\}^2 \times \mathbb{Z}^2_*, \end{cases} \tag{3.8}$$

$$\Diamond^{(1)}_{\overrightarrow{i}}(t) = \begin{cases} 0, & \overrightarrow{i} \in \mathbb{Z}^4_* \times \{(0,0)\}, \\ u_{\overrightarrow{i}-\overrightarrow{e}_5}(1,t), & \overrightarrow{i} \in \mathbb{Z}^4_* \times \{m \in \mathbb{Z}_+ : m \geq 1\} \times \{0\}, \\ u_{\overrightarrow{i}-\overrightarrow{e}_6}(0,t), & \overrightarrow{i} \in \mathbb{Z}^4_* \times \{0\} \times \{m \in \mathbb{Z}_+ : m \geq 1\}, \\ u_{\overrightarrow{i}-\overrightarrow{e}_5}(0,t) + u_{\overrightarrow{i}-\overrightarrow{e}_6}(1,t), & \overrightarrow{i} \in \mathbb{Z}^4_* \times \{m \in \mathbb{Z}_+ : m \geq 1\}^2. \end{cases} \tag{3.9}$$

In (3.7)–(3.9), the set $\{\overrightarrow{e}_l\}_{l=\overline{1,6}}$ is a family of unit vectors in $\mathbb{R}^6$; and the values $\left[u_{\overrightarrow{i}-\overrightarrow{e}_1}\right]_r$ and $\left[\frac{\partial u_{\overrightarrow{i}-\overrightarrow{e}_2}}{\partial t}\right]_r$ are interpreted as in Lemma 2.

4. A low-frequency asymptotic expansion

Theorem. *On account of the main assumptions and the technical conditions on the given data, problem (1.1)-(1.6) has a unique solution in* $\mathbb{H}^2(\Omega_T)$. *Additionally, if* $G, H \in C^n([-M, M])$ *for* n *given and* $(n, M) \in \mathbb{Z}_+ \times \mathbb{R}_+$, *then this unique strong solution possesses a low-frequency asymptotic expansion with respect to the six parameters* K, λ, a_0, c_1, a_1, *and* c_0 *up to an order* $n+1$ *as follows:*

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial t} \left(u - \sum_{0 \leq |\overrightarrow{i}| \leq n} u_{\overrightarrow{i}} \overrightarrow{\eta}^{\overrightarrow{i}} \right) \right\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \left\| u - \sum_{0 \leq |\overrightarrow{i}| \leq n} u_{\overrightarrow{i}} \overrightarrow{\eta}^{\overrightarrow{i}} \right\|_{L^\infty(0,T;H^1(\Omega))} \\
&+ \left\| \frac{\partial}{\partial t} \left(u(0,\cdot) - \sum_{0 \leq |\overrightarrow{i}| \leq n} u_{\overrightarrow{i}}(0,\cdot) \overrightarrow{\eta}^{\overrightarrow{i}} \right) \right\|_{L^2(0,T)} \\
&+ \left\| \frac{\partial}{\partial t} \left(u(1,\cdot) - \sum_{0 \leq |\overrightarrow{i}| \leq n} u_{\overrightarrow{i}}(1,\cdot) \overrightarrow{\eta}^{\overrightarrow{i}} \right) \right\|_{L^2(0,T)} \\
&\leq C_{ae} \left(K^2 + \lambda^2 + a_0^2 + c_1^2 + a_1^2 + c_0^2 \right)^{\frac{n+1}{2}} \qquad (4.1)
\end{aligned}
$$

for $\overrightarrow{\eta} = (K, \lambda, a_0, c_1, a_1, c_0)$ *and* $\overrightarrow{i} = (i_1, i_2, i_3, i_4, i_5, i_6) \in \mathbb{Z}_*^6$*; where* C_{ae} *is a non-negative constant independent of the six parameters* K*,* λ*,* a_0*,* c_1*,* a_1 *and* c_0*; and* $u_{\overrightarrow{i}}$ *is a unique strong solution of the problem* $\left(\mathbb{P}_{\overrightarrow{i}}\right)$ *defined by (3.1)–(3.9).*

Proof. For the existence, see [*J. Math. Phys.* **51** (2010) 103504, 28 pp]. To prove the asymptotic expansion result, there are three steps needed:

- ▷ Constructing three asymptotic error functions,
- ▷ Estimating asymptotic error functions,
- ▷ The asymptotic expansion for problem (1.1)-(1.6).

All details can be found in

Lê, U.V.: A semi-linear wave equation with space-time dependent coefficients and a memory boundary-like antiperiodic condition: a low-frequency asymptotic expansion. *J. Math. Phys.* **52**, 023510 (2011), 23 pp.

□

Thank you very much
for your attention

NOTE: “Alt+left arrow” for the previous view (hold Alt key and then press left arrow), similarly for “Alt+right arrow” for the next view.