New Characterizations of Besov-Triebel-Lizorkin-Hausdorff Spaces via Coorbits

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Joint work with Yiyu Liang, Yoshihiro Sawano, Dachun Yang, and Wen Yuan

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- **W. Yuan, W. Sickel and D. Yang**, Morrey and Campanato Meet Besov, Lizorkin and Triebel, **Lecture Notes in Math. 2005, Springer-Verlag, Berlin, 2010.**

Prof. Triebel (2002)...

Der Raum L^1 wird in seiner Häßlichkeit nur von L^∞ übertroffen.

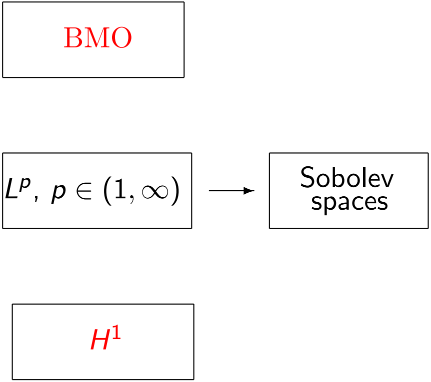
“Nice” function spaces

BMO

$L^p,\ p \in (1, \infty)$

H^1

“Nice” Function spaces



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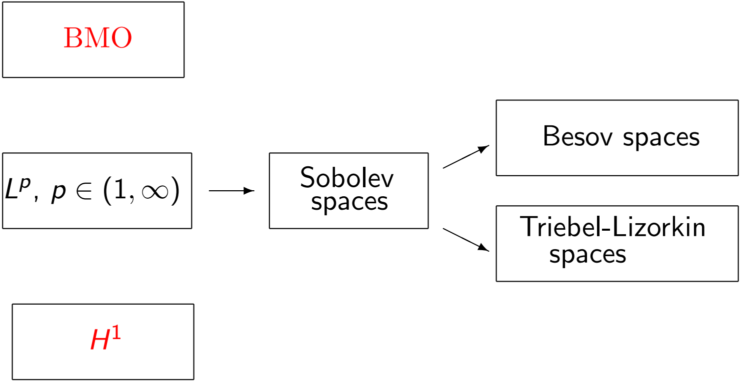

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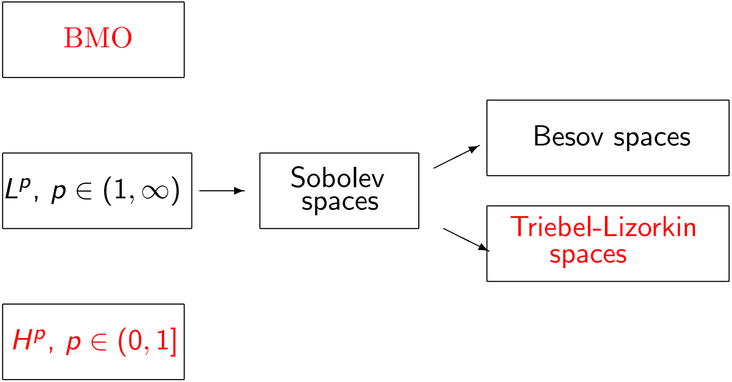

Homogeneous spaces

- $\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) x^\gamma \, dx = 0 \ \forall \ \gamma \in (\mathbb{N} \cup \{0\})^n \right\}$.

- Let
  $(*_1)$ $\varphi \in \mathcal{S}(\mathbb{R}^n)$;
  $(*_2)$ $\operatorname{supp} \hat{\varphi} \subset \left\{ \xi \in \mathbb{R}^n : \quad \frac{1}{2} \le |\xi| \le 2 \right\}$;
  $(*_3)$ when $\frac{3}{5} \le |\xi| \le \frac{5}{3}$, $|\hat{\varphi}(\xi)| \ge C > 0$.

- For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_j(x) = 2^{jn} \varphi(2^j x)$.

- Let $\mathcal{Q}(\mathbb{R}^n)$ be the set of all dyadic cubes in $\mathbb{R}^n$. For $P \in \mathcal{Q}(\mathbb{R}^n)$, let $j_P \equiv -\log_2 l(P)$.

Triebel-Lizorkin spaces

- (i) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. The **Triebel-Lizorkin space** $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} < \infty$, where when $p < \infty$,

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \equiv \left\| \left\{ \sum_{j\in\mathbb{Z}} (2^{js}|\varphi_j * f|)^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty;$$

and when $p = \infty$,

$$\|f\|_{\dot{F}^s_{\infty,q}(\mathbb{R}^n)} \equiv \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{1/q}} \left\{ \int_P \sum_{j=j_P}^{\infty} (2^{js}|\varphi_j * f(x)|)^q \, dx \right\}^{\frac{1}{q}} < \infty.$$

- (ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $0 \leq \tau < \infty$.

  The **Triebel-Lizorkin-type space** $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} (2^{js} |\varphi_j * f(x)|)^q \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty.$$

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- If $\tau = 0$, then $\dot{F}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{p,q}(\mathbb{R}^n)$.
- If $\tau \in (-\infty, 0)$, then $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}^n) = \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials in $\mathbb{R}^n$.

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- Frazier/Jawerth JFA 1990, Corollary 5.7: For all $p \in (0, \infty)$ we have $\dot{F}^{s,1/p}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{\infty,q}(\mathbb{R}^n)$.

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- This scale contains Morrey-Triebel-Lizorkin spaces and Q_α-spaces.

Triebel-Lizorkin-Hausdorff spaces

- (iii) Let $1 < p < \infty$, $s \in \mathbb{R}$, $1 < q < \infty$, $0 \le \tau \le \max\{p,q\}'$. The **Triebel-Lizorkin-Hausdorff space** $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ $(q \neq 1)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\| \left(\sum_{j\in\mathbb{Z}}^{\infty} \left(2^{js}|\varphi_j * f(x)|[\omega(x,2^{-j})]^{-1}\right)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

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The infimum in (i) and (ii) are taken over all nonnegative Borel functions ω on $\mathbb{R}^{n+1}_+$ such that

$$\int_{\mathbb{R}^n} [\sup_{|y-x|<t} |\omega(y,t)|]^{(\max\{p,q\})'} \, dH^{n\tau(\max\{p,q\})'}(x) \le 1. \qquad (1)$$

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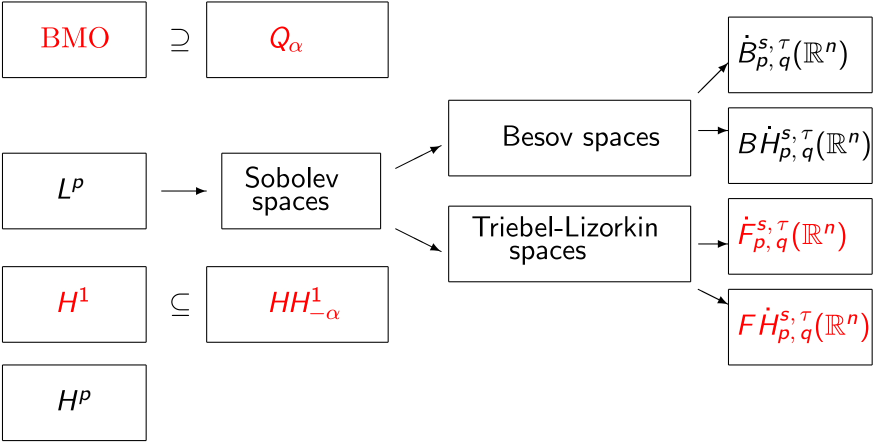
- If $\tau = 0$ then $F\dot{H}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{p,q}(\mathbb{R}^n)$
- $Q_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n)$, $HH^1_{-\alpha}(\mathbb{R}^n) = F\dot{H}^{-\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n)$
- **duality relation**: $[F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)]^* = \dot{F}^{-s,\tau}_{p',q'}(\mathbb{R}^n)$

More function spaces



Universal approach: coorbit space theory

- **Feichtinger/Gröchenig** 1980s, **Fornasier/Rauhut** 2007, **Rauhut/Ullrich** 2010
- $\mathcal{H}$...Hilbert space
- G...locally compact group with Haar measure μ
- Y...quasi-Banach function space on G, left-right translation invariant
- π...unitary group representation on $\mathcal{H}$;

$$x, y \in G \quad \Rightarrow \quad \pi(xy) = \pi(x) \circ \pi(y)$$

- Fix $g \in \mathcal{H}$, voice-transform of $f \in \mathcal{H}$

$$V_g f(x) = \langle f, \pi(x) g \rangle \quad , \quad x \in G$$

- Coorbit space $\text{Co}Y$ "given by"

$$f \in \text{Co}Y \quad \Longleftrightarrow \quad V_g f \in Y$$

Examples

- **Continuous Wavelet Transform**
  - $\mathcal{H} = L_2(\mathbb{R}^d)$, $G = ax + b$-group

$$(x,t)(y,s) = (x+ty, ts) \quad , \quad x,y \in \mathbb{R}^d \ , \ t,s > 0$$

  - Unitary representation π

$$\pi(x,t)g = T_x D_t g = t^{-d/2} g\Big(\frac{\cdot - x}{t}\Big)$$

- **Short Time Fourier Transform**
  - $\mathcal{G} = \mathbb{H}_d := \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ Heisenberg-group

$$(x,\omega,\tau)(x',\omega',\tau') = (x+x', w+w', \tau\tau' e^{\pi i x'\cdot\omega})$$

  - Schrödinger representation ρ

$$\rho(x,\omega,\tau)g := \tau e^{\pi i x\cdot\omega} T_x M_\omega g = \tau e^{-\pi i x\cdot\omega} e^{2\pi i\omega\cdot} g(\cdot - x)$$

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Besov-Triebel-Lizorkin spaces as coorbits

- [U10]**Peetre-type spaces on $\mathcal{G}$**
  For $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$ and $a \in (0,\infty)$,
  $$\|F\|_{\dot{P}^s_{p,q}(\mathcal{G})} = \left\| \left\{ \int_0^\infty t^{-sq} \sup_{z \in \mathbb{R}^n} \left[\frac{|F(\cdot + z, t)|}{(1 + |z|/t)^a} \right]^q \frac{dt}{t^{n+1}} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$
- $Y = \dot{P}^s_{p,q}(\mathcal{G})$ is left and right translation invariant
- **Theorem**
  For $p,\ q \in [1,\infty]$, $s \in \mathbb{R}$ and $a \in (n,\infty)$,
  $$\dot{F}^s_{p,\,q}(\mathbb{R}^n) = Co\dot{P}^{s+n(1/2-1/q)}_{p,q}.$$
- [U10] T. Ullrich, Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpolations as coorbits, J. Funct. Spaces Appl., to appear.

Besov-Lizorkin-Triebel-type spaces as coorbits

- Define the corresponding Peetre spaces $Y = \dot{P}^{s,\tau}_{p,q,a}$ by using the modified maximal function

$$\sup_{\substack{y\in\mathbb{R}^n \\ t/2\le r\le t}} \frac{|F(x+y,r)|}{(1+|y|/r)^a}$$

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- Indeed, $Y = \dot{P}^{s,\tau}_{p,q,a}$ is the space of all functions F such that

$$\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\int_0^{\ell(P)} t^{-sq} \left(\sup_{\substack{y\in\mathbb{R}^n \\ t/2\le r\le t}} \frac{|F(x+y,r)|}{(1+|y|/r)^a} \right)^q \frac{dt}{t^{n+1}} \right]^{p/q} dx \right\}^{1/p}$$

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- [LUSYY] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, Submitted.

- **Theorem** [LUSYY]

  (i) Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $q \in (0,\infty]$, $p \in (0,\infty)$, and $a \in (\frac{n}{\min\{p,q\}},\infty)$, then

$$\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = Co(\dot{P}^{s+n/2-n/q,\tau}_{p,q,a}(\mathcal{G})).$$

  (ii) If $p,q \in (1,\infty)$, $\tau \in [0,\frac{1}{(\max\{p,q\})'}]$, and $a \in (n(\frac{1}{\min\{p,q\}}+\tau),\infty)$, then

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- **Proof**: Continuous characterzations, translation invariance of $\dot{P}^{s,\tau}_{p,q,a}$, and coorbit space theory for quasi-Banach spaces developed in

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  (ii) If $p,q \in (1,\infty)$, $\tau \in [0,\frac{1}{(\max\{p,q\})'}]$, and $a \in (n(\frac{1}{\min\{p,q\}}+\tau),\infty)$, then

$$F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n) = Co(P\dot{H}^{s+n/2-n/q,\tau}_{p,q,a}(\mathcal{G})).$$

- **Proof**: Continuous characterzations, translation invariance of $\dot{P}^{s,\tau}_{p,q,a}$, and coorbit space theory for quasi-Banach spaces developed in
- [R07] H. Rauhut, Coorbit space theory for quasi-Banach spaces, Studia Math. 180 (2007), 237-253;

- **Theorem** [LUSYY]
  Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $p \in (0,\infty)$, $q \in (0,\infty]$, $R \in \mathbb{Z}_+ \cup \{-1\}$ and $a \in (\max\{n/p, n/q\}, \infty)$ such that $s + n\tau < R + 1$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the Tauberian condition. Then for all $f \in \mathcal{S}'_R(\mathbb{R}^n)$,

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} \sim \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\int_0^{\ell(P)} t^{-sq} |\varphi_t * f(x)|^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\int_0^{\ell(P)} t^{-sq} |(\varphi_t^* f)_a(x)|^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\int_0^{\ell(P)} t^{-sq} \int_{|z|<t} |\varphi_t * f(x+z)|^q \, dz \frac{dt}{t^{n+1}} \right)^{\frac{p}{q}} \right.$$

An abstract discretization result

Theorem (Rauhut/Ullrich 2010)

- $\mathcal{F} = \{\varphi_x\}_{x \in X}$
- Let $\mathcal{G}_r = \{\psi_x^r\}_{x \in X}, \tilde{\mathcal{G}}_r = \{\tilde{\psi}_x^r\}_{x \in X} \subset \mathcal{H}_v^1$ be continuous frames, and

$$K_r(x,y) = \sup_{z \in Q_x} \langle \varphi_y, \psi_z^r \rangle \quad , \quad \tilde{K}_r(x,y) = \sup_{z \in Q_x} \langle \varphi_y, \tilde{\psi}_z^r \rangle$$

  define bounded operators from Y to Y.
- Assume further that the expansion

$$f = \sum_{r=1}^{n} \sum_{i \in I} \langle f, \psi_{x_i}^r \rangle \tilde{\psi}_{x_i}^r \tag{2}$$

  holds true in $\mathcal{H}$. Then (2) extends to $\mathrm{Co}Y$. In particular

$$f \in \mathrm{Co}Y \quad \Longleftrightarrow \quad \{\langle f, \psi_{x_i}^r \rangle\}_{i \in I} \in Y^\sharp$$

Theorem [LUSYY]

- $(\psi^0, \tilde{\psi}^0), (\psi^1, \tilde{\psi}^1)$ generate a biorthogonal wavelet basis.
- $\psi^0, \tilde{\psi}^0$ satisfy (D), (S_K), and $\psi^1, \tilde{\psi}^1$ satisfy (D), (S_K), (M_{L-1}) with

$$K, L > M(p, q, s, \tau) \tag{3}$$

Application: biorthogonal wavelet bases

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Then every $f \in \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ has the decomposition

$$f = \sum_{c \in E \setminus \{0\}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \lambda^c_{j,k} 2^{\frac{jd}{2}} \tilde{\psi}^c(2^j \cdot -k) \tag{4}$$

where the sequences $\lambda^c = \{\lambda^c_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ defined by

$$\lambda^c_{j,k} = \langle f, 2^{\frac{jd}{2}} \psi^c(2^j \cdot -k) \rangle \quad , \quad j \in \mathbb{Z}, k \in \mathbb{Z}^d \,,$$

belong to the sequence space $(\dot{F}^{s,\tau}_{p,q})^\sharp$ for every $c \in E$.

Discussion

- New (discretization) results for special cases of our setting

  $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $q \in (0,\infty]$,
  $\mathrm{BMO}(\mathbb{R}^n)$ when $s = 0$, $q = 2$,
  The Q-space $Q_\alpha(\mathbb{R}^n)$,
  Hardy-Hausdorff space $HH_{-\alpha}(\mathbb{R}^n)$ for $\alpha \in (0, \min\{\frac{n}{2}, 1\})$,
  Triebel-Lizorkin-Morrey space $\dot{\mathcal{E}}^s_{upq}(\mathbb{R}^n)$ for $0 < p \leq u < \infty$, $s \in \mathbb{R}$

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- Conditions on wavelets just sufficient, not optimal
- **To do:** inhomogeneous spaces

Last slide

Thank you for your attention!