Traces of the weighted Sobolev spaces

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Tasks which are closely connected with description of the traces of the functional spaces $W_p^s(G)$ appeared in 30-s years 20th century after Sobolev's papers. They appeared like a new approach to solving some boundary problems for the elliptic partial differential equations.

First results about characterization of the traces of the functions from the Sobolev space $W_2^1(G)$ were given by Aronshain in 1955.

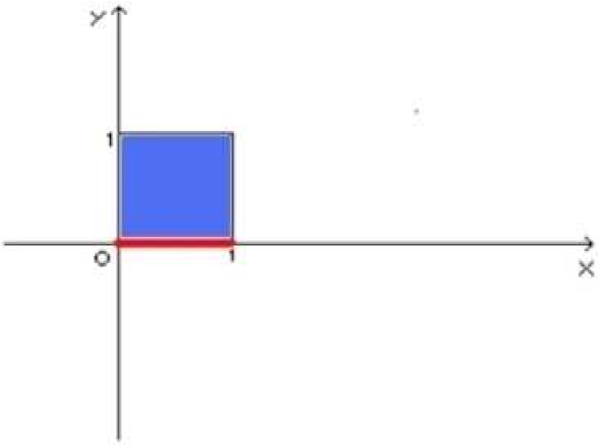
This question for the Sobolev spaces $W_p^1(G), 1 < p < \infty$ was proved by E.Gagliardo in 1957.

In 60-s years 20th century P.I. Lizorkin and S.V. Uspenskiy solved problem of characterization of the traces of functions from the weighted Sobolev spaces $W_p^1(G,\rho^r), G \subset R^n$.

(here function ρ^r - distance to the boundary in some degree).

These results were generalized on a case of Sobolev spaces with higher smoothness and for $p=2$ applied to the problems of elliptic equations with degeneration on the boundary of the domain(S.M. Nikol'skiy, P.I. Lizorkin).

In 1979 G.A. Kalyabin solved trace problem for the weighted Sobolev spaces. In his paper the weight functions depended only on the transverse variable.



Let $\beta : Q^{n-1} \to R_+$ - measurable on $Q^{n-1}(x_n = 0)$ weight function.

Consider the weighted Sobolev space $W_p^1(Q^n, \beta)(1 < p < \infty)$, which is the normed space of all functions f, such that the weak derivatives are D_{x_i} measurable on Q^n and

$$\|f\|_{W_p^1(Q^n,\beta)} = \left(\iint_{Q^n} \beta(x') \left| f(x', x_n) \right|^p dx' dx_n \right)^{\frac{1}{p}} + $$

$$+ \sum_{i=1}^{n} \left(\iint_{Q^n} \beta(x') \left| D_{x_i} f(x', x_n) \right|^p dx' dx_n \right)^{\frac{1}{p}} < +\infty.$$

We research problem of characterization of the traces functions from the weighted Sobolev spaces $W_p^1(Q^n, \beta)(1 < p < \infty)$ on the face $Q^{n-1}(x_n = 0)$.

We consider weighted functions which belong to the different classes and traces are characterized in terms of belonging to the weighted Besov spaces $B_p^{1-\frac{1}{p}}(Q^{n-1}, \beta)$, which are constucted in present paper.

For functions $f \in W_p^1(Q^2, x^r), -1 < r < p-1$ we obtain trace characterization on the whole boundary ∂Q^2 of the square Q^2.

Also we solve Dirichlet problem for elliptic partial equations with degeneration on the part of the boundary.

Trace theorems

Let $U_{1,k}$- set of all positive functions β_k , measurable on $Q^{n-1}(x_n=0)$, which monotonically decrease for every variable $x_1,\ldots,x_k$ at fixed others, monotonically increase for every variable $x_{k+1},\ldots,x_{n-1}$ at fixed others and

$$\int_0^1 \beta_k(x_1,\ldots,x_j,\ldots,x_{n-1})dx_j=+\infty, 1\le j\le k$$

Let $B_p^{1-\frac{1}{p}}(Q^{n-1},\beta_k)$ - the normed space of all functions $\varphi\in L_p^{loc}(Q^{n-1})$ with the following norm

$$\|\varphi\|_{B_p^{1-\frac{1}{p}}}=\left[\int_{Q^{n-1}}\beta_k(x')|\varphi(x')|^p\,dx'\right]^{\frac{1}{p}}+\left[\int_{Q^{n-1}}\int_{Q^{n-1}}\beta_k(x')\frac{|\varphi(x'+h)-\varphi(x')|^p}{|h|^{p-1}}dx'\frac{dh}{|h|^{n-1}}\right]^{\frac{1}{p}}.$$

Theorem1. Let $1<p<\infty$, then $Tr\big|_{Q^{n-1}(x_n=0)}W_p^1(Q^n,\beta_k)=B_p^{1-\frac{1}{p}}(Q^{n-1},\beta_k)$

Let U_z - set of all positive measurable on $(0,1]$ functions σ, which monotonically decrease on $(0,1]$ and

$$\frac{1}{y}\int_0^y \sigma(t)dt \le C(\sigma)\sigma(y), y \in (0,1].$$

Example: $\sigma(x) = x^{-\alpha}, 0 \le \alpha < 1.$

$B_p^{1-\frac{1}{p}}((0,1),\sigma)$ - the normed space of all functions $\varphi \in L_p^{loc}((0,1))$ with the following norm

$$\|\varphi\|_{B_p^{1-\frac{1}{p}}((0,1),\sigma)} = \left[\int_0^1 \sigma(t)\,|\varphi(t)|^p\, dt\right]^{\frac{1}{p}} + \left[\int_0^1 \sigma(t)\int_0^t \frac{|\varphi(t-h)-\varphi(t)|^p}{|h|^{p-1}}\frac{dh}{|h|}\,dt\right]^{\frac{1}{p}}$$

Theorem2. Let $1 < p < \infty$, then $Tr\big|_{x_2=0} W_p^1(Q^2,\sigma) = B_p^{1-\frac{1}{p}}((0,1),\sigma).$

Let U_3- set of all positive measurable on $(0,1]$ functions γ, such that:

1) $\int_0^1 \gamma(t)dt < \infty$

2) for all $r \in [0,1)$ there exists constant $C(r) > 0$, such that

$$\frac{\gamma(x)x^r}{\gamma(t)t^r} \le C(r), 0 < t \le x \le 1.$$

Example: $\gamma(x) = \dfrac{1}{x\left|\ln \dfrac{x}{2}\right|^2}$.

$B_p^{1-\frac{1}{p}}((0,1),\gamma)$- the normed space of all functions $\varphi \in L_p^{loc}((0,1))$, which have 0 as the Lebesgue point, with the following norm:

$$\|\varphi\|_{E_p^{1-\frac{1}{p}}((0,1),\gamma)} = \left[\int_0^1 \gamma(t)|\varphi(t)|^p\, dt\right]^{\frac{1}{p}} + \left[\int_0^1 \gamma(t)\int_0^t \frac{|\varphi(t-h)-\varphi(t)|^p}{|h|^{p-1}}\frac{dh}{|h|}dt\right]^{\frac{1}{p}} + \left[\int_0^1 \gamma(t)\frac{|\varphi(t)-\varphi(0)|^p}{t^{p-1}}\, dt\right]^{\frac{1}{p}}.$$

Theorem3. Let $1 < p < \infty$, then $Tr|_{x_2=0} W_p^1(Q^2,\gamma) = B_p^{1-\frac{1}{p}}((0,1),\gamma)$.

TRACE ON THE WHOLE BOUNDARY AND THE JUNCTION CONDITIONS

Theorem 4. Let $f \in W_p^1(Q^2, x^r), -1 < r < p-1$, then there exists trace of the function f on ∂Q^2, in sense $\varphi_j = Tr|_{\Gamma_j} f, j = 1,2,3,4.$ In addition

$$\varphi_1, \varphi_3 \in B_p^{1-\frac{1}{p}}((0,1), x^r); \varphi_2 \in B_p^{1-\frac{1+r}{p}}((0,1)); \varphi_4 \in B_p^{1-\frac{1}{p}}((0,1))(1)$$

and also:

$$\|\varphi_1\|_{B_p^{1-\frac{1}{p}}((0,1),x^r)} + \|\varphi_3\|_{B_p^{1-\frac{1}{p}}((0,1),x^r)} + \|\varphi_2\|_{B_p^{1-\frac{1+r}{p}}((0,1))} +$$

$$+ \|\varphi_4\|_{B_p^{1-\frac{1}{p}}((0,1))} \le C_2 \|f\|_{W_p^1(Q^2,x^r)}$$

$$\sum_{i=1}^{3} \iint_{Q^2} \frac{x^r}{x+y} |\varphi_{i+1}(y) - \varphi_i(x)|^p \, dxdy + \iint_{Q^2} \frac{x^r}{x+y} |\varphi_1(y) - \varphi_4(x)|^p \, dxdy \le C_3 \|f\|_{W_p^1(Q^2,x^r)}.$$

Constants C_2, C_3 don't depend on f.

Theorem 5. Let condition (1) hold and

$$\sum_{i=1}^{3}\iint_{Q^2}\frac{x^r}{x+y}\left|\varphi_{i+1}(y)-\varphi_i(x)\right|^p dxdy+\iint_{Q^2}\frac{x^r}{x+y}\left|\varphi_1(y)-\varphi_4(x)\right|^p dxdy<+\infty.$$

Then there exists a function $f\in W_p^1(Q^2,x^r)$, s. t. $\varphi_j=Tr|_{\Gamma_j} f, j=1,2,3,4$ and

$$\|f\|_{W_p^1(Q^2,x^r)}\le C_4\Bigg(\|\varphi_1\|_{B_p^{1-\frac{1}{p}}((0,1),x^r)}+\|\varphi_3\|_{B_p^{1-\frac{1}{p}}((0,1),x^r)}+\|\varphi_2\|_{B_p^{1-\frac{1+r}{p}}((0,1))}+\|\varphi_4\|_{B_p^{1-\frac{1}{p}}((0,1))}+$$

$$+\sum_{i=1}^{3}\iint_{Q^2}\frac{x^r}{x+y}\left|\varphi_{i+1}(y)-\varphi_i(x)\right|^p dxdy+\iint_{Q^2}\frac{x^r}{x+y}\left|\varphi_1(y)-\varphi_4(x)\right|^p dxdy\Bigg).$$

APPLICATIONS

Let on the square Q^2 are given measurable functions $a_{\alpha\beta}$ with the next properties:

$$a_{\alpha\beta}(x,y) = a_{\beta\alpha}(x,y), |a_{\alpha\beta}(x,y)| \le C_1 x^r, -1 < r < 1,$$

$$\sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}(x,y)\xi_\alpha \xi_\beta \ge \kappa x^r \sum_{|\alpha|=1} (\xi_\alpha)^2, (x,y) \in Q^2.$$

We will consider two cases:

$$a) F \in L_2(Q^2), 0 \le r < 1,$$

$$b) x^{-\frac{r}{2}} F \in L_2(Q^2), -1 < r < 0.$$

Consider in cases a) or b) problem

$$\begin{cases} Lu = F, Lu = \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\beta|} D^\beta (a_{\alpha\beta} D^\alpha u) (1), \\ Tr|_{\partial Q}\, u = \{\varphi_j\}_{j=1}^4 (2). \end{cases}$$

Theorem 6. If the cases either a) or b) take place and conditions of theorem 5 hold, then the solution of the Dirichlet problem (1),(2) exists and is unique.

INCREASING SMOTHNESS OF THE WEAK SOLUTIONS

Let some number $l \in \mathbb{N}$ be fixed, consider next conditions:

- $$\left\| x^{1-\frac{r}{2}+|\lambda|} D^{\lambda} F \right\|_{L_2(Q^2)} < +\infty, |\lambda| \le l-1, (3)$$

  $$\left| D^{\lambda} a_{\alpha\beta}(x,y) \right| \le \frac{cx^{r}}{x^{|\lambda|}}, |\lambda| \le l. (4)$$

Now consider for any fixed $s \in \mathbb{R}$ the normed space $W_2^{1+l}(Q^2, x^s)$, which consists of all functions $f \in L_2^{loc}(Q^2)$, such that

$$\|f\|_{W_2^{1+l}(Q^2,x^s)} := \|f\|_{L_2(Q^2)} + \sum_{|\alpha|=l+1} \left\| x^s D^{\alpha} f \right\|_{L_2(Q^2)} < +\infty.$$

Theorem 7. Let the cases a) or b) take place, let conditions (3) ,(4) and conditions of the theorem 5 hold, then solution of the Dirichlet problem (1),(2)

- $U \in W_2^{1+l}(Q^2, x^{r+2l})$.