The sharp A_p constant between BMO and weighted BMO

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Our Problem

It is well known that for $w \in A_\infty$,

$$BMO = BMO(w),$$

where

$$\|f\|_{BMO(w)} := \sup_Q \inf_{c \in \mathbb{C}} \frac{1}{w(Q)} \int_Q |f - c| w dx.$$

Our purpose is to investigate estimates of the ration $\|f\|_{BMO(w)} / \|f\|_{BMO}$ from above and below with A_p constant, $[w]_{A_p}$.
That is, we want to find $\Phi\ \Psi : [1, \infty) \to (0, \infty)$ satisfying

$$c_{n,p} \Phi([w]_{A_p}) \leq \frac{\|f\|_{BMO(w)}}{\|f\|_{BMO}} \leq c_{n,p} \Psi([w]_{A_p}).$$

Notations

- w:weight $\iff$ $0 \le w \in L^1_{loc}$ and $w(Q) := \int_Q w dx$.
- $\langle f \rangle_Q := \frac{1}{|Q|} \int_Q f dx$ and $\langle f \rangle_{Q,w} := \frac{1}{w(Q)} \int_Q f w dx$.
- T: Calderón-Zygmund operator.
- $w \in A_p$ $\iff$

$$[w]_{A_p} := \sup_Q \begin{cases} \langle w \rangle_Q \, \|w^{-1}\|_{L^\infty(Q)}, \text{ if } p = 1 \\ \langle w \rangle_Q \, \langle w^{1-p'} \rangle_Q^{p-1}, \text{ if } p \in (1,\infty) \\ \langle w \rangle_Q \, \exp(\langle \log w^{-1} \rangle_Q), \text{ if } p = \infty \end{cases} < \infty$$

$$[w]_{A_\infty} \le [w]_{A_p} \le [w]_{A_1} \text{ and } A_\infty = \cup_p A_p.$$

- $T : L^p(w) \to L^p(w); bdd \iff w \in A_p$.

Notations

- M. Wilson introduced another A_∞ constant;

$$[w]'_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M_Q w dx,$$

where

$$M_Q f(x) := \sup_{x \in R \subset Q} \frac{1}{|R|} \int_R |f| dy.$$

$$[w]'_{A_\infty} \le c_n [w]_{A_\infty}.$$

and

$$[w]'_{A_\infty} < \infty \iff w \in A_\infty.$$

The sharp weighted inequalities

- Recently, many authors are interested to find the optimal order of α_p in the weighted inequality;

$$\|T\|_{L^p(w)\to L^p(w)} \le c_{n,T,p}[w]_{A_p}^{\alpha_p}.$$

Hytönen proved that $\alpha_p = \max(1, 1/(p-1))$ is the optimal order, i.e.

$$\|T\|_{L^p(w)\to L^p(w)} \le c_{n,p,T}[w]_{A_p}^{\max(1,1/(p-1))}.$$

- In the case $p = 1$, Lerner-Ombrosi-Pérez proved

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \le c_{n,T}[w]_{A_1}(1+\log[w]_{A_1}).$$

However, we do not know whether this order is optimal or not.

The upper bound

The optimal upper bound of $\|f\|_{BMO(w)}/\|f\|_{BMO}$ was found by Hytönen and Pérez. They showed

$$\|f\|_{BMO(w)} \lesssim [w]'_{A_\infty}\|f\|_{BMO}. \tag{1}$$

(1) has an application to the sharp weighted inequality;

$$\|T\|_{L^\infty(w)\to BMO(w)} \lesssim [w]'_{A_\infty}. \tag{2}$$

The upper bound

To prove the upper bound (1), Hytönen and Pérez used the reverse Hölder inequality;

$$\langle w^{r_w} \rangle_Q^{1/r_w} \leq 2 \langle w \rangle_Q,$$

where

$$r_w := 1 + (c_n [w]'_{A_\infty})^{-1} \in (1, 2).$$

Using the equivalence form of the definition of $[w]'_{A_\infty}$, we give another proof of (1) without the reverse Hölder inequality.

John-Nirenberg inequality

We recall a version of John-Nirenberg inequality due to Mateu-Mattila-Nicolau-Orobitg; for any non-negative Radon measure w which satisfies $w(L) = 0$ for all hyperplanes L, orthogonal to one of the coordinate axes, we have

$$w(\{x \in Q; |f - \langle f \rangle_{Q,w}| > \lambda\}) \leq c_n w(Q) \exp\left(-\frac{\lambda}{c_n \|f\|_{BMO(w)}}\right), \tag{3}$$

for all cubes Q and $\lambda > 0$, where c_n is independent of w.

Then, in particular, for $w \in A_\infty$, (3) holds.

John-Nirenberg inequality

Also, (3) yields the equivalence

$$\sup_Q \|f - \langle f\rangle_{Q,w}\|_{\exp L(Q;w)} \approx \|f\|_{BMO(w)}, \tag{4}$$

where the implicit constants are independent of w,

$$\|f\|_{\exp L(Q)} := \inf\Big\{\lambda > 0; \frac{1}{|Q|}\int_Q (\exp(\frac{|f|}{\lambda}) - 1)dx \le 1\Big\},$$

Another proof of the upper bound

To give the equivalence form of $[w]'_{A_\infty}$, we recall $L\log L$-theory by E. Stein; for any cube Q,

$$\|f\|_{L\log L(Q)} \approx \langle |f| \log(e + \frac{|f|}{\langle |f| \rangle_Q}) \rangle_Q \approx \langle M_Q f \rangle_Q,$$

where $\|f\|_{L\log L(Q)} := \inf\Big\{\lambda > 0; \frac{1}{|Q|}\int_Q \frac{|f|}{\lambda} \log(e + \frac{|f|}{\lambda}) dx \le 1\Big\}$. Then,

$$\begin{aligned} [w]'_{A_\infty} &:= \sup_Q \frac{1}{w(Q)} \int_Q M_Q w dx \\ &\approx \sup_Q \frac{1}{w(Q)} \int_Q w \log(e + \frac{w}{\langle w \rangle_Q}) dx \\ &\approx \sup_Q \frac{1}{\langle w \rangle_Q} \|w\|_{L\log L(Q)}. \end{aligned}$$

Another proof of the upper bound

By using a Hölder inequality;

$$\langle |fg| \rangle_Q \leq c_n \|f\|_{\exp L(Q)} \|g\|_{L \log L(Q)},$$

we have

$$\begin{aligned}
\inf_{c \in \mathbb{C}} \langle |f - c| \rangle_{Q,w} &\leq \langle |f - \langle f \rangle_Q| \rangle_{Q,w} \\
&= \frac{|Q|}{w(Q)} \langle |f - \langle f \rangle_Q| w \rangle_Q \\
&\leq c_n \frac{1}{\langle w \rangle_Q} \|f - \langle f \rangle_Q\|_{\exp L(Q)} \|w\|_{L \log L(Q)} \\
&\leq c_n [w]'_{A_\infty} \|f\|_{BMO}.
\end{aligned}$$

$$\|f\|_{BMO(w)} \leq c_n [w]'_{A_\infty} \|f\|_{BMO}.$$

The characterization of $[w]'_{A_\infty}$

In the proof above, we used

$$\langle f\rangle_{Q,w} \lesssim [w]'_{A_\infty}\|f\|_{\exp L(Q)}.$$

From the $\exp L(Q)$ – $L\log L(Q)$ duality, we can find $g \in \exp L(Q)$ such that

$$\|g\|_{\exp L(Q)}\|w\|_{L\log L(Q)} \le c_n\langle g\rangle_{Q,w}\langle w\rangle_Q.$$

In the end, we get

$$[w]'_{A_\infty} \approx \sup_{Q,f}\frac{\langle |f|\rangle_{Q,w}}{\|f\|_{\exp L(Q)}}.$$

The lower bound

Next, we consider the lower bound of $\|f\|_{BMO(w)}/\|f\|_{BMO}$;

The lower estimate

$$\|f\|_{BMO} \lesssim \log(2[w]_{A_\infty})\|f\|_{BMO(w)}. \quad (5)$$

- We do not the similar estimate with $[w]'_{A_\infty}$ and whether the order is optimal or not.
- If

  $$\|f\|_{BMO} \lesssim \|f\|_{BMO(w)}$$

  is true, then the order ”0” is optimal.

  Let $w = t\chi_E + \chi_{E^c}$, $(t \gg 1,\ |E| < \infty)$. Then, $[w]_{A_\infty} \approx \dfrac{t}{\log t}$ and

  $$\|\log w\|_{BMO} = \|\log w\|_{BMO(w)} = \frac{1}{2}\log t.$$

The lower bound

We begin the proof with

$$\frac{1}{w(Q)}\int_Q\Big(\exp\big(\frac{|f-\langle f\rangle_{Q,w}|}{\|f-\langle f\rangle_{Q,w}\|_{\exp L(Q;w)}}\big)-1\Big)w\,dx \le 1,$$

which is equal to

$$\langle\exp\big(\frac{|f-\langle f\rangle_{Q,w}|}{\|f-\langle f\rangle_{Q,w}\|_{\exp L(Q;w)}}\big)\rangle_{Q,w} \le 2.$$

By using the inequality

$$\exp(\langle g\rangle_Q) \le [w]_{A_\infty}\langle\exp(g)\rangle_{Q,w}$$

with $g = \dfrac{|f-\langle f\rangle_{Q,w}|}{\|f-\langle f\rangle_{Q,w}\|_{\exp L(Q;w)}}$,

The lower bound

one obtains

$$\exp\Big(\langle \frac{|f - \langle f\rangle_{Q,w}|}{\|f - \langle f\rangle_{Q,w}\|_{\exp L(Q;w)}}\rangle_Q\Big) \le 2[w]_{A_\infty},$$

that is,

$$\langle |f - \langle f\rangle_{Q;w}|\rangle_Q \le \log(2[w]_{A_\infty})\|f - \langle f\rangle_{Q,w}\|_{\exp L(Q;w)}.$$

Then, from (4), we have

$$\begin{aligned}
\|f\|_{BMO} &\le \sup_Q \langle |f - \langle f\rangle_{Q,w}|\rangle_Q \\
&\le \log(2[w]_{A_\infty}) \sup_Q \|f - \langle f\rangle_{Q,w}\|_{\exp L(Q;w)} \\
&\lesssim \log(2[w]_{A_\infty})\|f\|_{BMO(w)}.
\end{aligned}$$