Dominating mixed smoothness, Faber bases, integration

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$$\varphi_0(t) = 1 \text{ if } |t| \leq 1 \quad \text{and} \quad \varphi_0(v) = 0 \text{ if } |v| \geq 3/2,$$
$$\varphi_m(t) = \varphi_0(2^{-m}t) - \varphi_0(2^{-m+1}t), \qquad m \in \mathbb{N}.$$

$n \in \mathbb{N}$, here $n = 2$. Rectangular tensor decomposition in $\mathbb{R}^2$:

$$\varphi_k(x) = \varphi_{k_1}(x_1)\varphi_{k_2}(x_2), \qquad k = (k_1, k_2) \in \mathbb{N}_0^2, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Definition 1. (i) $0 < p, q \leq \infty$, $r \in \mathbb{R}$. Then $S^r_{pq}B(\mathbb{R}^2)$: All $f \in S'(\mathbb{R}^2)$ such that

$$\|f \,|S^r_{pq}B(\mathbb{R}^2)\|_\varphi = \Big(\sum_{k \in \mathbb{N}_0^2} 2^{r(k_1+k_2)q} \big\|(\varphi_k \widehat{f})^\vee \,|L_p(\mathbb{R}^2)\big\|^q \Big)^{1/q} < \infty$$

(with the usual modification if $q = \infty$).

(ii) $r \in \mathbb{N}_0$, $1 < p < \infty$. Then $S^r_p W(\mathbb{R}^2)$: All $f \in S'(\mathbb{R}^2)$ such that

$$\|f \,|S^r_p W(\mathbb{R}^2)\| = \sum_{\beta \in \mathbb{N}_0^2; 0 \leq \beta_1, \beta_2 \leq r} \|D^\beta f \,|L_p(\mathbb{R}^2)\| < \infty.$$

Remark 2. F-counterpart,

$$\|f\,|S^1_pW(\mathbb{R}^2)\| = \|f\,|L_p(\mathbb{R}^2)\| + \sum_{m=1}^{2} \Big\|\frac{\partial f}{\partial x_m}\,|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\,|L_p(\mathbb{R}^2)\Big\|.$$

Long history: Russian school, derivatives, differences, $\sim$ 1960, Babenko 60, Bakhvalov 59/64, Nikol'skij 62/63, Korobov 59.
Fourier-analytical approach: Schmeisser 80/82, $S^r_{pq}B(\mathbb{R}^n)$, $S^r_{pq}F(\mathbb{R}^n)$, Bazarkhanov 03,
Building blocks (atoms, wavelets): Vybíral 06.

$$n = 1: \quad S^r_{pq}B(\mathbb{R}) = B^r_{pq}(\mathbb{R}), \quad S^r_pW(\mathbb{R}) = W^r_p(\mathbb{R}).$$

Definition 3. $Q = \{x \in \mathbb{R}^2 : \ 0 < x_1 < 1, \ 0 < x_2 < 1\}$ square in $\mathbb{R}^2$. Then $S^r_{pq}B(Q)$ is the restriction of $S^r_{pq}B(\mathbb{R}^2)$ to Q. Similarly $S^r_pW(Q)$.

Remark 4. Intrinsic norms, $1 < p < \infty$:

$$\|f\,|S^1_pW(Q)\| \sim \|f\,|L_p(Q)\| + \sum_{m=1}^{2} \Big\|\frac{\partial f}{\partial x_m}\,|L_p(Q)\Big\| + \Big\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\,|L_p(Q)\Big\|.$$

Similarly $S^r_pW(Q)$, $r \in \mathbb{N}$, $1 < p < \infty$. Also for $S^r_{pp}B(Q)$, $r > 0$, $1 \le p \le \infty$ with mixed intrinsic differences. Extension problem.

[T10]: Bases in function spaces, sampling, discrepancy, numerical integration (book, 2010). Aim of talk: several extensions, in particular to weighted spaces.

[T12]: Faber systems and their use in sampling, discrepancy, numerical integration (European Math. Soc. Lecture Notes)

$$w^\alpha(x) = (1+x_1^2)^{\alpha/2}(1+x_2^2)^{\alpha/2}, \quad x=(x_1,x_2) \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}.$$

Definition 5. $0 < p,q \le \infty$, $r \in \mathbb{R}$, $\alpha \in \mathbb{R}$. $S^r_{pq}B(\mathbb{R}^2,\alpha)$: All $f \in S'(\mathbb{R}^2)$ with

$$\|f \,|S^r_{pq}B(\mathbb{R}^2,\alpha)\| = \|w^\alpha f \,|S^r_{pq}B(\mathbb{R}^2)\| < \infty.$$

Similarly F-spaces and weighted Sobolelv spaces $S^r_pW(\mathbb{R}^2,\alpha)$ with $r \in \mathbb{N}_0$, $1 < p < \infty$.

Remark 6. Dominating mixed smoothness counterpart to weighted isotropic spaces $A^s_{pq}(\mathbb{R}^n,\alpha)$,

$$\|f \,|A^s_{pq}(\mathbb{R}^n,\alpha)\| = \|w_\alpha f \,|A^s_{pq}(\mathbb{R}^n)\|$$

where $w_\alpha(x) = (1+|x|^2)^{\alpha/2}$, $x \in \mathbb{R}^n$. Has some history, equivalent quasi-norms.

$$\|f\,|L_p(\mathbb{R}^2, w^\alpha)\| = \Big(\int_{\mathbb{R}^2} w^\alpha(x)^p |f(x)|^p \,\mathrm{d}x\Big)^{1/p}, \qquad 0 < p \le \infty.$$

Proposition 7. r, p, q as above, $\alpha \in \mathbb{R}$. Then

$$\|f\,|S^r_{pq}B(\mathbb{R}^2, \alpha)\| \sim \Big(\sum_{k \in \mathbb{N}_0^2} 2^{r(k_1+k_2)q} \big\|(\varphi_k \widehat{f})^\vee\,|L_p(\mathbb{R}^2, w^\alpha)\big\|^q\Big)^{1/q}$$

(equivalent quasi-norms). If $r \in \mathbb{N}_0$, $1 < p < \infty$ then

$$\|f\,|S^r_pW(\mathbb{R}^2, \alpha)\| \sim \sum_{\beta \in \mathbb{N}_0; 0 \le \beta_1, \beta_2 \le r} \|D^\beta f\,|L_p(\mathbb{R}^2, w^\alpha)\|.$$

(equivalent norms).

Proof: As in [ET96] (based on Haroske-T, 94/95) for $A^s_{pq}(\mathbb{R}^n, \alpha)$: lifts, local means, based on kernels with product structure.

$$x \in \mathbb{R}, \qquad v_k(x) = \left(1 - 2^{k+1}|x|\right)_+, \qquad k \in \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}.$$

Faber functions:

$$v_{-1,m}(x) = v_{-1}(x - m), \qquad m \in \mathbb{Z},$$
$$v_{km}(x) = v_k\left(x - 2^{-k-1} - 2^{-k}m\right), \qquad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}.$$

Basic space of continuous functions:

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \; f(x) \to 0 \text{ if } |x| \to \infty\},$$

usual sup-norm. Differences:

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad \Delta_h^{m+1} f = \Delta_h^m \Delta_h^1 f, \quad m \in \mathbb{N}.$$
$$-\tfrac{1}{2}\left(\Delta^2_{2^{-k-1}} f\right)(2^{-k}m) = f\left(2^{-k}m + 2^{-k-1}\right) - \tfrac{1}{2} f\left(2^{-k}m\right) - \tfrac{1}{2} f\left(2^{-k}m + 2^{-k}\right).$$

Proposition 8.

$$\{v_{km}: \ k \in \mathbb{N}_{-1}, \ m \in \mathbb{Z}\} \quad \text{conditional basis in } C_0(\mathbb{R}),$$

$$f(x) = \sum_{K=0}^{\infty} \big(f_{K+1}(x) - f_K(x) \big) + f_0(x), \qquad x \in \mathbb{R},$$

with

$$f_K(x) = \sum_{|m| \le K} f(m) v_{-1,m}(x) - \tfrac{1}{2} \sum_{0 \le k \le K; |m| \le K} \big(\Delta^2_{2^{-k-1}} f \big)(2^{-k} m) v_{km}(x).$$

Definition 9. $w^\alpha(x) = w_\alpha(x) = (1+x^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$. Then $b_{pq}(\mathbb{R}, \alpha)$ collects all

$$\mu = \big\{ \mu_{km} \in \mathbb{C}: \ k \in \mathbb{N}_{-1}, \ m \in \mathbb{Z} \big\} \subset \mathbb{C},$$

with

$$\| \mu \, | b_{pq}(\mathbb{R}, \alpha) \| = \Big(\sum_{k \in \mathbb{N}_{-1}} \Big(\sum_{m \in \mathbb{Z}} w^\alpha (2^{-k} m)^p |\mu_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

Theorem 10. $0 < p, q \le \infty$, $\frac{1}{p} < r < 1 + \min(1, \frac{1}{p})$, $\alpha > 0$. Let $f \in S'(\mathbb{R})$. Then $f \in B^r_{pq}(\mathbb{R}, \alpha)$ if, and only if,

$$f = \sum_{k \in \mathbb{N}_{-1}} \sum_{m \in \mathbb{Z}} \mu_{km} \, 2^{-k(r-\frac{1}{p})} v_{km}, \qquad \mu \in b_{pq}(\mathbb{R}, \alpha),$$

unconditional convergence in $C_0(\mathbb{R})$. The representation is unique,

$$\mu_{-1,m}(f) = f(m), \quad m \in \mathbb{Z},$$
$$\mu_{km}(f) = -2^{k(r-\frac{1}{p})-1} \big(\Delta^2_{2^{-k-1}} f \big)(2^{-k} m), \qquad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}.$$

Furthermore, $f \mapsto \mu(f)$ is an isomorphic map. If $\max(p,q) < \infty$ then $\{v_{km}\}$ unconditional basis.

Remark 11. Main point: $\mu_{km}(f)$ evaluates f at 3 points. Starting point for *numerical integration*. Recall

$$B^r_{pq}(\mathbb{R}, \alpha) \hookrightarrow B^r_{pq}(\mathbb{R}) \hookrightarrow C(\mathbb{R}), \qquad r > 1/p.$$

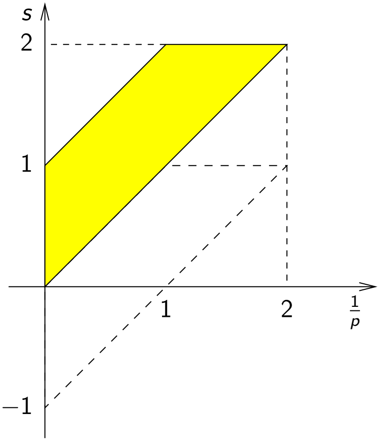

Figure: Faber bases, $r = s$

$x = (x_1, x_2) \in \mathbb{R}^2$, $\mathbb{N}^2_{-1} = \mathbb{N}_{-1} \times \mathbb{N}_{-1}$. Tensor product of 1-dim. Faber system:

$$\left\{ v_{km}(x) = v_{k_1,m_1}(x_1) v_{k_2,m_2}(x_2), \quad k = (k_1, k_2) \in \mathbb{N}^2_{-1}, \quad m = (m_1, m_2) \in \mathbb{Z}^2 \right\}.$$

(Mixed) differences:

$$\Delta^2_{h,1} f(x_1, x_2) = f(x_1 + 2h, x_2) - 2f(x_1 + h, x_2) + f(x_1, x_2)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $h \in \mathbb{R}$. Similarly $\Delta^2_{h,2} f(x_1, x_2)$,

$$\Delta^{2,2}_{h_1,h_2} f(x_1, x_2) = \Delta^2_{h_2,2}\left(\Delta^2_{h_1,1} f\right)(x_1, x_2), \qquad h_1 \in \mathbb{R}, \quad h_2 \in \mathbb{R}.$$

Then

$$\begin{aligned}
d^2_{km}(f) &= f(m) \quad \text{if } k = (-1,-1),\ m \in \mathbb{Z}^2, \\
d^2_{km}(f) &= -\tfrac{1}{2} \Delta^2_{2^{-k_2-1},2} f(m_1, 2^{-k_2} m_2) \text{ if } k = (-1, k_2),\ k_2 \in \mathbb{N}_0,\ m \in \mathbb{Z}^2, \\
d^2_{km}(f) &= -\tfrac{1}{2} \Delta^2_{2^{-k_1-1},1} f(2^{-k_1} m_1, m_2) \text{ if } k = (k_1, -1),\ k_1 \in \mathbb{N}_0,\ m \in \mathbb{Z}^2, \\
d^2_{km}(f) &= \tfrac{1}{4} \Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f(2^{-k_1} m_1, 2^{-k_2} m_2) \text{ if } k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2.
\end{aligned}$$

$$C_0(\mathbb{R}^2) = \left\{ f \in C^2(\mathbb{R}^2) : \ f(x) \to 0 \text{ if } |x| \to \infty \right\}.$$

Proposition 12. $\{v_{km}: \; k \in \mathbb{N}^2_{-1}, \; m \in \mathbb{Z}^2\}$ is a conditional basis in $C_0(\mathbb{R}^2)$,

$$f(x) = \sum_{K=0}^{\infty} \big(f_{K+1}(x) - f_K(x) \big) + f_0(x), \qquad f \in C_0(\mathbb{R}^2),$$

convergence in $C_0(\mathbb{R}^2)$, with

$$f_K(x) = \sum_{\substack{k \in \mathbb{N}^2_{-1} \\ k_1 \le K, \, k_2 \le K}} \; \sum_{\substack{m \in \mathbb{Z}^2 \\ |m_1| \le K, \, |m_2| \le K}} d^2_{km}(f) \, v_{km}(x), \quad K \in \mathbb{N}_0.$$

Definition 13. $w^\alpha(x) = (1 + x_1^2)^{\alpha/2} (1 + x_2^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$. Then $s_{pq}b(\mathbb{R}^2, \alpha)$ is the collection of all

$$\mu = \big\{ \mu_{km} \in \mathbb{C}: \; k \in \mathbb{N}^2_{-1}, \; m \in \mathbb{Z}^2 \big\}$$

with

$$\| \mu \, | s_{pq}b(\mathbb{R}^2, \alpha) \| = \Big(\sum_{k \in \mathbb{N}^2_{-1}} \Big(\sum_{m \in \mathbb{Z}^2} w^\alpha \big(2^{-k_1} m_1, 2^{-k_2} m_2 \big)^p |\mu_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty,$$

(usual modification if $\max(p, q) = \infty$).

Theorem 14. Either $p = q = \infty$ or $0 < p, q < \infty$. Let $\frac{1}{p} < r < 1 + \min(\frac{1}{p}, 1)$ and $\alpha > 0$. Let $f \in S'(\mathbb{R}^2)$. Then $f \in S^r_{pq}B(\mathbb{R}^2, \alpha)$ if, and only if,

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, v_{km}, \qquad \mu \in s_{pq}b(\mathbb{R}^2, \alpha),$$

unconditional convergence being in $C_0(\mathbb{R}^2)$. The representation is unique,

$$\mu_{km} = \mu_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}\, d^2_{km}(f), \qquad k \in \mathbb{N}^2_{-1}, \quad m \in \mathbb{Z}^2.$$

Furthermore, $J: \ f \mapsto \mu(f)$ is an isomorphic map of $S^r_{pq}B(\mathbb{R}^2, \alpha)$ onto $s_{pq}b(\mathbb{R}^2, \alpha)$. If $p < \infty$, $q < \infty$, then

$$\left\{ 2^{-(k_1+k_2)(r-\frac{1}{p})}\, w^{\alpha}\big(2^{-k_1}m_1, 2^{-k_2}m_2\big)^{-1}\, v_{km}: \ k \in \mathbb{N}^2_{-1}, \ m \in \mathbb{Z}^2 \right\}$$

is a unconditional (normalised) basis in $S^r_{pq}B(\mathbb{R}^2, \alpha)$.
Remark 15. $d^2_{km}(f)$ evaluates f in at most 9 points, basis for numerical integration.

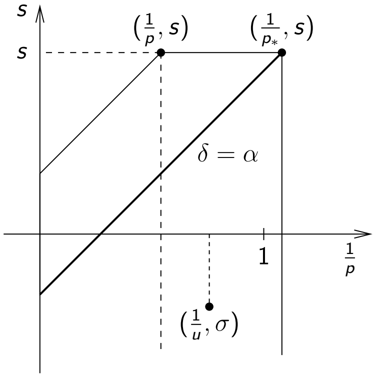

Figure: Weighted spaces, $\frac{1}{p_*} = \frac{1}{p} + \alpha$, $s = r$

Numerical integration in

$$\mathbb{R}: \quad \int_{\mathbb{R}} f(x)\mathrm{d}x, \quad f \in B^r_{pq}(\mathbb{R}, \alpha) \hookrightarrow L_1(\mathbb{R}) \cap C(\mathbb{R}),$$

$$\mathbb{R}^2: \quad \int_{\mathbb{R}^2} f(x)\,\mathrm{d}x, \quad S^r_{pq}B(\mathbb{R}^2, \alpha) \hookrightarrow L_1(\mathbb{R}^2) \cap C(\mathbb{R}^2).$$

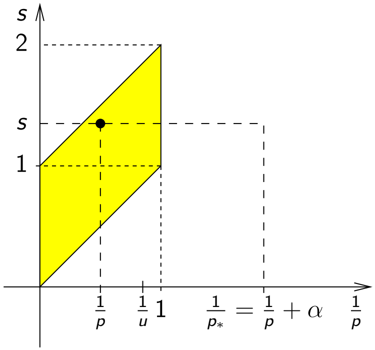

Figure: Integration, simplest case, $r = s$

Definition 16. Let $1 \le p \le \infty$, $0 < q \le \infty$, $\alpha > 1 - \frac{1}{p}$, $r > \frac{1}{p}$. Then

$$\mathrm{Int}_k\left(S^r_{pq}B(\mathbb{R}^2,\alpha)\right) = \inf\left[\sup\left\{\left|\int_{\mathbb{R}^2} f(x)\mathrm{d}x - \sum_{j=1}^{k} a_j\, f(x^j)\right| : f \in S^r_{pq}B(\mathbb{R}^2,\alpha),\ \|f\,|S^r_{pq}B(\mathbb{R}^2,\alpha)\| \le 1\right\}\right]$$

where the infimum is taken over all $\{x^j\}_{j=1}^k \subset \mathbb{R}^2$ and all $\{a_j\}_{j=1}^k \subset \mathbb{C}$. Similarly

$$\mathrm{Int}_k\left(S^r_{pq}B(\mathbb{R}^2)\right)_\alpha = \inf\left[\sup\left\{\left|\int_{\mathbb{R}^2} f(x)w^{-\alpha}(x)\,\mathrm{d}x - \sum_{j=1}^{k} a_j f(x^j)\right| : f \in S^r_{pq}B(\mathbb{R}^2),\ \|f\,|S^r_{pq}B(\mathbb{R}^2)\| \le 1\right\}\right],$$

where again the infimum is taken over all $\{x^j\}_{j=1}^k \subset \mathbb{R}^2$ and all $\{a_j\}_{j=1}^k \subset \mathbb{C}$.

Remark 17. Same for $S^r_pW(\mathbb{R}^2,\alpha)$ and $S^r_pW(\mathbb{R}^2)$. Similarly if $n \ge 3$. If $n = 1$ then $B^r_{pq}(\mathbb{R},\alpha)$ etc.

Theorem 18. Let $1 \le p \le \infty$, $\alpha > 1 - \frac{1}{p}$, $\frac{1}{p} < r < 1 + \frac{1}{p}$.
(i) In addition $\alpha > r + 1 - \frac{1}{p}$. Then for $2 \le k \in \mathbb{N}$,

$$c_1\, k^{-r} (\log k)^{1-\frac{1}{p}} \le \mathrm{Int}_k \big(S^r_{pp} B(\mathbb{R}^2, \alpha) \big) \le c_2\, k^{-r} (\log k)^{r+1-\frac{1}{p}}.$$

Same for $\mathrm{Int}_k \big(S^r_{pp} B(\mathbb{R}^2) \big)_\alpha$.
(ii) In addition $\alpha < r + 1 - \frac{1}{p}$, $0 < q \le \infty$. Then for $2 \le k \in \mathbb{N}$,

$$c\, k^{-\alpha+1-\frac{1}{p}} (\log k)^{\alpha} \le \mathrm{Int}_k \big(S^r_{pq} B(\mathbb{R}^2, \alpha) \big) \le c_\varepsilon\, k^{-\alpha+1-\frac{1}{p}} (\log k)^{2\alpha+\varepsilon}.$$

Same for $\mathrm{Int}_k \big(S^r_{pq} B(\mathbb{R}^2) \big)_\alpha$.
Proof. Basic idea: Splitting of

$$f = \sum_{k,m} \underbrace{d^2_{km}(f)\, v_{km}}_{\text{at most 9 function values of } f} = \underbrace{\sum{}' \cdots}_{\text{finite sum}} + \sum{}'' \cdots .$$

Corollary 19. Let $1 < p < \infty$, $\alpha > 1 - \frac{1}{p}$.

(i) In addition $p \geq 2$, $\alpha > 2 - \frac{1}{p}$. Then for $2 \leq k \in \mathbb{N}$,

$$c_1 k^{-1} (\log k)^{1/2} \leq \mathrm{Int}_k \big(S^1_p W(\mathbb{R}^2, \alpha) \big) \leq c_2 \, k^{-1} (\log k)^{2 - \frac{1}{p}}.$$

(ii) In addition $\alpha < 2 - \frac{1}{p}$. Then for $2 \leq k \in \mathbb{N}$,

$$c \, k^{-\alpha + 1 - \frac{1}{p}} (\log k)^{\alpha} \leq \mathrm{Int}_k \big(S^1_p W(\mathbb{R}^2, \alpha) \big) \leq c_\varepsilon \, k^{-\alpha + 1 - \frac{1}{p}} (\log k)^{2\alpha + \varepsilon}.$$

Same for $\mathrm{Int}_k \big(S^1_p W(\mathbb{R}^2) \big)_\alpha$.

Proof of (ii) by embeddings of $S^1_p W(\mathbb{R}^2)$ between $S^r_{pq} B(\mathbb{R}^2)$ spaces, also right-hand side of part (i) where $p = q$ in part (i) of Theorem 18 causes now $p \geq 2$.

1. If $n \in \mathbb{N}$ then one has to replace $\log k$ in Theorem 18, Corollary 19, by $(\log k)^{n-1}$. Same conditions, same breaking points. If $n = 1$ then usual spaces B^r_{pq} and W^1_p: equivalences.

2. Breaking point for α same as in isotropic case for $A^s_{pq}(\mathbb{R}^n, w_\alpha)$, here with $n = 1$.

3. Restriction $r > \frac{1}{p}$ natural (embedding in $C(\mathbb{R}^2)$). Restriction $r < 1 + \frac{1}{p}$ comes from use of above Faber bases. Extension to $r \geq 1 + \frac{1}{p}$? Possible. Use Faber splines. So far only $n = 1$, not so final, first steps, [T12].

4. Sampling numbers g_k, g_k^{lin} : Replace $L_1(\mathbb{R}^2)$ by $L_u(\mathbb{R}^2)$. So far again simplest case: $u \leq p$, $\frac{1}{p_*} = \frac{1}{p} + \alpha > \frac{1}{u}$.

5. Problems: Other cases, limiting situations, breaking points. Other numbers: approximation numbers, entropy numbers etc.

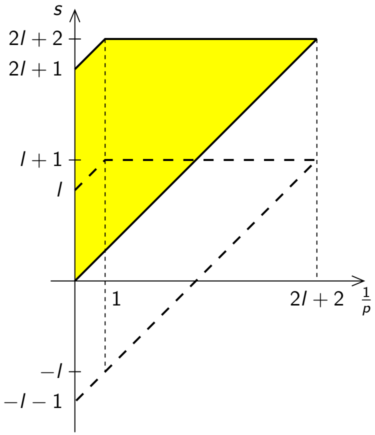

Figure: Faber splines, $l \in \mathbb{N}_0$