A theoretical study of a free supercritical flow with superficial tension.

C. Titri-Bouadjenak,
R. Ait-Yahia, D. Hernane,D. Teniou.
AMNEDP Laboratory, Mathematics Faculty,
U.S.T.H.B, Bab-Ezzouar, ALGIERS, ALGERIA

23 septembre 2011

presented by C. TITRI-BOUADJENAK,

FSDONA, Tabarz/Thür(Germany), September 18-24, 2011

TALK PLAN

Introduction

Position of the problem

Transformation of the domain and governing equations.

Mathematics tools

Existence and uniqueness result

Introduction

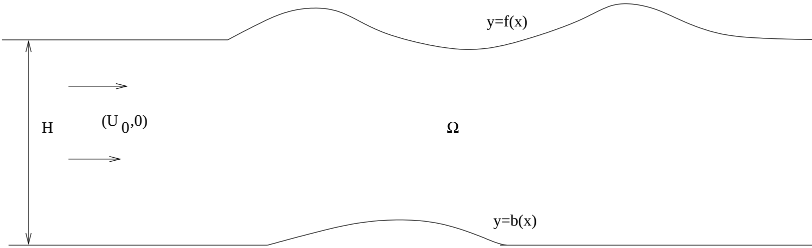

Fig1

Flow :stationary, irrotational. Fluid : Inviscid, incompressible. gravity , superficial tension.

- $y = b(x)$, the bottom equation ; $y = H + \gamma(x)$, the free surface equation,
  b is $C^2(\mathbb{R})$, compact support,
- $\Omega_b^\gamma = \{(x, y) \in \mathbb{R}^2; b(x) < y < H + \gamma(x)\}$, the fluid domain,
- $\overrightarrow{\nu}$ the exterior normal to the boundary of the domain Ω_b^γ,
- $\overrightarrow{u}(x, y)$ the velocity fluid ; ρ its density assumed to be constant,
- $C(x)$ the curvature of the free surface given by

$$C(x) = \frac{-\gamma''(x)}{[1 + (\gamma'(x))^2]^{\frac{3}{2}}}$$

and σ the superficial tension coefficient.

Position of the problem

$$div\,\overrightarrow{u} = 0 \quad in \quad \Omega_b^\gamma \tag{1}$$

$$curl\,\overrightarrow{u} = 0 \quad in \quad \Omega_b^\gamma \tag{2}$$

$$\overrightarrow{u}.\overrightarrow{\nu} = 0 \quad on \quad y = H + \gamma(x) \tag{3}$$

$$\overrightarrow{u}.\overrightarrow{\nu} = 0 \quad on \quad y = b(x) \tag{4}$$

$$\lim_{x \to \pm\infty} \overrightarrow{u} = (U_0, 0) \tag{5}$$

$$\lim_{x \to \pm\infty} \gamma(x) = 0 \tag{6}$$

$$\frac{\rho}{2}|\overrightarrow{u}|^2 + \rho g y + \sigma C(x) = cst \quad on \quad y = H + \gamma(x) \tag{7}$$

Nondimensionalization $\longrightarrow$ Froude number $F = \frac{U_0}{\sqrt{gH}}$
new $\Omega_b^\gamma = \{(x, y) \in \mathbb{R}^2; b(x) < y < 1 + \gamma(x)\}$
Incompressible fluid , irrotational flow $\Rightarrow \exists \psi$ such that :

$$\overrightarrow{u}(x, y) = (\frac{\partial \psi}{\partial y}(x, y), -\frac{\partial \psi}{\partial x}(x, y))$$

The bottom and the free surface of Ω_b^γ being streamlines, ψ is constant on $y = b(x)$ and on $y = 1 + \gamma(x)$. We choose $\psi = 0$ on the bottom, then $\psi = 1$ on the free surface.

Governing equations

$$\triangle\psi = 0 \ in \ \Omega_b^\gamma \tag{8}$$

$$\psi = 0 \ on \ y = b(x) \tag{9}$$

$$\psi = 1 \ on \ y = 1 + \gamma(x) \tag{10}$$

$$\lim_{x\to\pm\infty} \psi(x,y) = y \tag{11}$$

$$\lim_{x\to\pm\infty} \gamma(x) = 0 \tag{12}$$

$$\frac{F^2}{2}|\overrightarrow{\nabla\psi}|^2 + 1 + \gamma(x) + \sigma C(x) = cste \ \ on \ y = 1 + \gamma(x) \tag{13}$$

Perturbation stream function $\Rightarrow \psi = y + \psi_p$

$$\triangle\psi_p = 0 \ in \ \Omega_b^\gamma \tag{14}$$

$$\psi_p = -b(x) \ on \ y = b(x) \tag{15}$$

$$\psi_p = -\gamma(x) \ on \ y = 1 + \gamma(x) \tag{16}$$

$$\lim_{x\to\pm\infty} \psi_p(x,y) = 0 \tag{17}$$

$$\lim_{x\to\pm\infty} \gamma(x) = 0 \tag{18}$$

$$\frac{F^2}{2}\{|\overrightarrow{\nabla}\psi_p|^2 + 2\frac{\partial\psi_p}{\partial y}\} + \gamma(x) + \sigma C(x) = 0 \ \ on \ y = 1 + \gamma(x) \tag{19}$$

Transformation of the domain.

$$\begin{cases} \tilde{x} & = & x \\ \tilde{y} & = & \frac{y-b(x)}{1+\gamma(x)-b(x)} \end{cases} \tag{20}$$

Ω_b^γ becomes

$$Q = \left\{(x,y) \in \mathbb{R}^2 / -\infty < x < +\infty,\ 0 < y < 1\right\}$$

$$\psi_p(x,y) = \overset{\sim}{\psi}_p(\tilde{x}, \tilde{y}) \tag{21}$$

Equations in the fix domain.

$$\begin{cases} \Delta\widetilde{\psi}_p + \mathcal{P}_b^{\gamma}\widetilde{\psi}_p = 0 \qquad in \;\; Q \\ \widetilde{\psi}_p(\widetilde{x},0) = -b(\widetilde{x}), \;\; \widetilde{x} \in \mathbb{R} \\ \widetilde{\psi}_p(\widetilde{x},1) = -\gamma(\widetilde{x}), \;\;\; \widetilde{x} \in \mathbb{R} \\ \lim\limits_{\widetilde{x}\to\pm\infty} \;\; \widetilde{\psi}_p(\widetilde{x},\widetilde{y}) = 0 \\ \lim\limits_{\widetilde{x}\to\pm\infty} \;\; \gamma(\widetilde{x}) = 0 \\ T(b,\gamma) := \sigma\gamma''(\widetilde{x}) - \frac{F^2}{2}(1+\gamma'^2(\widetilde{x}))^{\frac{3}{2}}[|\widetilde{\nabla}_{b,\gamma}\widetilde{\psi}_p|^2(\widetilde{x},1) + \\ + \frac{2}{1+\gamma-b}\frac{\partial\widetilde{\psi}_p}{\partial\widetilde{y}}(\widetilde{x},1)] - (1+\gamma'^2(\widetilde{x}))^{\frac{3}{2}}\gamma(\widetilde{x}) = 0 \end{cases} \tag{22}$$

$$\widetilde{\nabla}_{b,\gamma} = \begin{pmatrix} \frac{\partial}{\partial \widetilde{x}} + \frac{-b' - \widetilde{y}(\gamma' - b')}{1+\gamma-b} \frac{\partial}{\partial \widetilde{y}} \\ \frac{1}{1+\gamma-b} \frac{\partial}{\partial \widetilde{y}} \end{pmatrix}$$

$$\mathcal{P}_b^\gamma = a_1 \frac{\partial^2}{\partial \widetilde{x} \partial \widetilde{y}} + a_2 \frac{\partial^2}{\partial \widetilde{y}^2} + a_3 \frac{\partial}{\partial \widetilde{y}}$$

$a_1 = 2\frac{\widetilde{y}(b'-\gamma')-b'}{1+\gamma-b}$;
$a_2 = (\frac{a_1}{2})^2 - 1 + \frac{1}{(1+\gamma-b)^2}$

$$a_3 = \frac{-1}{1+\gamma-b}[b'' + \widetilde{y}(\gamma'' - b'')] +$$

$$+ \frac{2}{(1+\gamma-b)^2}(\gamma' - b')[b' + \widetilde{y}(\gamma' - b')]$$

Mathematics tools : The spaces

for $m \in \mathbb{N}$, $\lambda \in]0,1[$ and $c > 0$, we introduce the following spaces :
i)The space $B_c^{m,\lambda}(\overline{Q})$ defined by :
$B_c^{m,\lambda}(\overline{Q}) =$

$$\left\{ v \in C^{m,\lambda}(\overline{Q}) / \sup_{k+l \leq m} \sup_{(\widetilde{x},\widetilde{y}) \in \overline{Q}} e^{c\left|\widetilde{x}\right|} \left| D^k_{\widetilde{x}} D^l_{\widetilde{y}} v(\widetilde{x},\widetilde{y}) \right| < \infty \right\}$$

provided with the norm :

$$\|v\|_{m,c,\lambda} = \sum_{k+l\leq m} \sup_{(\widetilde{x},\widetilde{y})\in\overline{Q}} e^{c\left|\widetilde{x}\right|} \left| D^k_{\widetilde{x}} D^l_{\widetilde{y}} v(\widetilde{x},\widetilde{y})\right| +$$

$$\sup_{k+l=m} \sup_{(\widetilde{x},\widetilde{y})\neq(\widetilde{x'},\widetilde{y'})} \frac{\left|D^k_{\widetilde{x}} D^l_{\widetilde{y}} v(\widetilde{x},\widetilde{y}) - D^k_{\widetilde{x}} D^l_{\widetilde{y}} v(\widetilde{x'},\widetilde{y'})\right|}{\left[(\widetilde{x}-\widetilde{x'})^2+(\widetilde{y}-\widetilde{y'})^2\right]^{\lambda/2}}$$

which is a Banach algebra.

ii) The space $B_c^{m,\lambda}(\mathbb{R})$ defined by :

$$B_c^{m,\lambda}(\mathbb{R}) = \left\{ v \in C^{m,\lambda}(\mathbb{R}) / \sum_{0 \leq k \leq m} \sup_{x \in \mathbb{R}} e^{c|x|} \left| D_x^k v(x) \right| < \infty \right\}$$

provided with the norm :

$$\|v\|_{m,c,\lambda} = \sum_{0 \leq k \leq m} \sup_{x \in \mathbb{R}} e^{c|x|} \left| D_x^k v \right| + \sup_{\substack{(x,x') \in \mathbb{R}^2 \\ x \neq x'}} \frac{|D_x^m v(x) - D_x^m v(x')|}{|x - x'|^\lambda}$$

which is also a Banach algebra.

Motivation choice of these spaces : Banach algebras ; asymptotic behavior of the solution. $(\widetilde{\psi}, \gamma)$ at infinity.

Existence and uniqueness result

Define $T : B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) \to B_c^{0,\lambda}(\mathbb{R})$
$T(b, \gamma)$ given by (22).
We have

$$T(0, 0) = 0$$

We apply the implicit function theorem to the equation

$$T(b, \gamma) = 0$$

Differentiability of T with respect to b and γ, theorem 1

Theorem : There exists $\widetilde{c} > 0$ such that for all $c \in]0, \widetilde{c}[$ and for all $\lambda \in]0, 1[$, there exists an open ball $\mathcal{B}$ of radius $r_0 > 0$, centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$ such that whenever $(b, \gamma) \in \mathcal{B}$, the following statements hold :
a)The problem :

$$\left\{ \begin{array}{rcll} \Delta\psi & = & 0 \text{ in } \Omega_b^\gamma & \\ \psi(x, 1+\gamma(x)) & = & -\gamma(x), & x \in \mathbb{R} \\ \psi(x, b(x)) & = & -b(x), & x \in \mathbb{R} \end{array} \right.$$

has a unique solution ψ such that, $\widetilde{\psi}$ the transform of ψ by (21), is in $B_c^{2,\lambda}(\overline{Q})$.

b) The mapping $S : (b, \gamma) \mapsto \widetilde{\psi}$ is continuously differentiable from $\mathcal{B}$ into $B_c^{2,\lambda}(\overline{Q})$.

Proposition 1

Proposition

Let be the boundary value problem

$$\left\{\begin{array}{rcl} \Delta v & = & b_1 \text{ in } Q \\ v(\tilde{x},1) & = & b_2(\tilde{x}),\ \tilde{x} \in \mathbb{R} \\ v(\tilde{x},0) & = & b_3(\tilde{x}),\ \tilde{x} \in \mathbb{R} \end{array}\right. \tag{23}$$

where $(b_1,b_2,b_3) \in B_c^{0,\lambda}(\overline{Q}) \times B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$,, then there exists $\tilde{c}$ such that whenever $0 < c < \tilde{c}$ problem (23) has a unique solution v in $B_c^{2,\lambda}(\overline{Q})$; furthermore, the solution map is a topological isomorphism between the corresponding spaces.

Lemma 1 Let $(b,\gamma) \in B_c^{2,\lambda}(\mathbb{R})) \times B_c^{2,\lambda}(\mathbb{R}))$. We have $(a_1,a_2,a_3) \in (B_c^{0,\lambda}(\overline{Q}))^3$, furthermore :

$$\|a_i\|_{B_c^{0,\lambda}(\overline{Q})} \leq L_i(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}), \ \ 1 \leq i \leq 3$$

where $L_i(.,.)$ are continuous functions verifying $L_i(0,0) = 0$.

Lemma 2 Let P be a rational function of k variables which is devoid of poles in a neighborhood of the origin in $\mathbb{R}^k$(k a positive integer), such that $P(0) = 0$. Then the mapping

$$P : \prod_{1\leq i\leq k} B_c^{n_i,\lambda}(\mathbb{R}) \longrightarrow B_c^{n_0,\lambda}(\mathbb{R})$$

$$(g_1, \ldots\ldots, g_k) \longmapsto P(g_1, \ldots\ldots, g_k)$$

where $n_i \in \mathbb{N},\ 1 \leq i \leq k,\ \ n_0 = \min\{n_i,\ 1 \leq i \leq k\}$, is continuously differentiable in a neighborhood of the origin in $\prod_{1\leq i\leq k} B_c^{n_i,\lambda}(\mathbb{R})$.

Proposition 2

Proposition The operator $T : B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) \to B_c^{0,\lambda}(\mathbb{R})$ defined by (22) is continuously Fréchet differentiable from $\mathcal{B}$ into $B_c^{0,\lambda}(\mathbb{R})$ where $\mathcal{B}$ is defined in Theorem 1.

Expression of $\frac{\partial T}{\partial \gamma}(0,0)$

We have : $\widetilde{\psi}$ is the solution of the problem :

$$\left\{\begin{array}{rcl} \Delta\widetilde{\psi} + \mathcal{P}_b^{\gamma}\widetilde{\psi} & = & 0 \ \text{ in } Q \\ \widetilde{\psi}(\widetilde{x},0) & = & -b(\widetilde{x}) \ , \quad \widetilde{x} \in \mathbb{R} \\ \widetilde{\psi}(\widetilde{x},1) & = & -\gamma(\widetilde{x}) \ , \quad \widetilde{x} \in \mathbb{R} \end{array}\right.$$

and $\widetilde{\psi}_{|b=\gamma=0} = 0$ in Q and $\mathcal{P}_0^0\widetilde{\psi} = 0$.
Let $h \in C_\eta^{2,\alpha}(\mathbb{R})$. We put $b = 0$ in the problem above, we derive with respect to γ in the direction h and we evaluate the derivative at $\gamma = 0(see[2])$. We denote by :

$$w_h \equiv \frac{\partial \widetilde{\psi}}{\partial \gamma}\,|_{b=\gamma=0} \ .h$$

Proposition 3

We obtain :

$$\frac{\partial T}{\partial \gamma}(0,0).h = \sigma h'' - h - F^2 \frac{\partial w_h}{\partial y}(.,1) \tag{24}$$

where w_h is the solution of the system of the following proposition.

Proposition Let $h \in C^{2,\alpha}_{\eta}(\mathbb{R})$; then w_h is the unique solution of the problem :

$$\left\{ \begin{array}{rcl} \Delta w & = & 0 \ \text{ in } Q \\ w(\widetilde{x},0) & = & 0, \ \widetilde{x} \in \mathbb{R} \\ w(\widetilde{x},1) & = & -h(\widetilde{x}), \ \widetilde{x} \in \mathbb{R} \end{array} \right. \tag{25}$$

Proof : Apply Prop1.

Invertibiliy of $\frac{\partial T}{\partial \gamma}(0,0)$: *theorem2*

The invertibiliy of the operator $\frac{\partial T}{\partial \gamma}(0,0)$ consists of proving that for a given q in $B_c^{0,\lambda}(\mathbb{R})$, there exists one and only one h in $B_c^{2,\lambda}(\mathbb{R})$ such that

$$\sigma h'' - h - F^2 \frac{\partial w_h}{\partial y}(.,1) = q \tag{26}$$

with w_h verifying the system (25).

Theorem : The operator $h \mapsto \sigma h'' - h - F^2 \frac{\partial w_h}{\partial y}(.,1)$ is an isomorphism from $B_c^{2,\lambda}(\mathbb{R})$ into $B_c^{0,\lambda}(\mathbb{R})$, where w_h is the solution of the system (25)

Proposition 4

To prove this theorem, we need

Proposition

Let f in $\mathbf{B}_c^{0,\lambda}(\mathbb{R})$ and consider the differential equation

$$v'' - \frac{1}{\sigma} v = f, \tag{27}$$

for $x \in \mathbb{R}$ and σ a positive constant.
(i) For every c with $0 < c < \frac{1}{\sqrt{\sigma}}$, and every f in $\mathbf{B}_c^{0,\lambda}(\mathbb{R})$, (27) has a unique solution v in $\mathbf{B}_c^{2,\lambda}(\mathbb{R})$.
(ii) There exist a positive constant α such that

$$\|v\|_{c,2,\lambda} \leq \alpha \|f\|_{c,0,\lambda}$$

Proof of theorem 2

To prove : for a given f in $B_c^{0,\lambda}(\mathbb{R})$, there exists a unique h in $B_c^{2,\lambda}(\mathbb{R})$ verifying :

$$\sigma h'' - h - F^2 \frac{\partial w_h}{\partial y} = f \tag{28}$$

where w_h is the solution of the boundary value problem (25).
Equation (28) is equivalent to :

$$h'' - \frac{1}{\sigma} h - \frac{F^2}{\sigma} \frac{\partial w_h}{\partial y} = \frac{1}{\sigma} f \tag{29}$$

Denoting by $P_\sigma(D)$ the operator $D_x^2 - \frac{1}{\sigma} Id$, we may write (29) in the form :

$$h - \frac{F^2}{\sigma} P_\sigma(D)^{-1} \frac{\partial w_h}{\partial y} = \frac{1}{\sigma} P_\sigma(D)^{-1} f$$

By proposition 4, we have :

$$\| P_\sigma(D)^{-1} \frac{\partial w_h}{\partial y} \|_{B_c^{2,\lambda}(\mathbb{R})} \leq \alpha \| \frac{\partial w_h}{\partial y} \|_{B_c^{1,\lambda}(\mathbb{R})} \leq \alpha \| w_h \|_{B_c^{2,\lambda}(\mathbb{R})},$$

and by proposition 1, we get :

$$\| P_\sigma(D)^{-1} \frac{\partial w_h}{\partial y} \|_{B_c^{2,\lambda}(\mathbb{R})} \leq \alpha \| w_h \|_{B_c^{2,\lambda}(\mathbb{R})} \leq \beta \| h \|_{B_c^{2,\lambda}(\mathbb{R})}$$

Then the operator defined from $B_c^{2,\lambda}(\mathbb{R})$ into $B_c^{2,\lambda}(\mathbb{R})$ by :

$$h \mapsto h - P_\sigma(D)^{-1}\frac{\partial w_h}{\partial y}$$

is invertible if $\frac{F^2\beta}{\sigma} < 1$, i.e $0 < F < F_0$, where $F_0 = \sqrt{\frac{\sigma}{\beta}}$
From this, we conclude that the operator :

$$\frac{\partial T}{\partial \gamma}(0,0) : B_c^{2,\lambda}(\mathbb{R}) \longrightarrow B_c^{0,\lambda}(\mathbb{R})$$

$$h \mapsto \frac{\partial T}{\partial \gamma}(0,0).h = \sigma h'' - h - F^2\frac{\partial w_h}{\partial y}$$

is an isomorphism.

Theorem : For all c, $0 < c < min(\widetilde{c}, \frac{1}{\sqrt{\sigma}})$, for all $\lambda \in]0,1[$, and for all $F \in]0, F_0[$, there exists an open ball $\mathcal{B}$ of radius $r_0 > 0$, centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, there exists a neighborhood $\mathcal{V}_b$ of zero in $B_c^{2,\lambda}(\mathbb{R})$, there exists a mapping $\varphi : \mathcal{V}_b \longrightarrow B_c^{2,\lambda}(\mathbb{R})$ of class $\mathcal{C}^1$, such that :
$\{\forall (b,\gamma) \in \mathcal{B}, T(b,\gamma) = 0\} \Leftrightarrow \{b \in \mathcal{V}_b, \gamma = \varphi(b)\}$.

[1] F. Helein, Ecoulement stationnaire dans un canal à fond presque plat, prépub. CMLA(1991), groupe d'hydrodynamique navale, URA CNRS 853. Univ Paris Diderot, Paris 7. UFR de Mathématiques.

[2] F. Abergel, J.L. Bona, A Mathematical Theory For Viscous, Free Surface Flows Over a Perturbed Plane,Arch. Rational Mech. Anal,118(1)(1992),71-93. Univ. Paris-Sud, lab. analyse numérique, 91405 Orsay,France.

9

- [3]C. Titri-Bouadjenak, D. Hernane-Boukari D,R. Ait Yahia-Djouadi, D. Teniou : Existence and Uniqueness of the Solution of a Supercritical Free Surface Flow Problem Over an Obstacle, published online in Revista Matemätica complutense,(8 April 2010). Univ. U.S.T.H.B,Faculté de Mathématiques,Lab. AMNEDP, Alger, Algérie.
- [4] D. Boukari ; R. Djouadi ; and D. Teniou : Free surface flow over an obstacle. Theoretical study of the fluvial case ; Abs. and Appl. Anal ; 6 :7, 413-429 (2001).