About problems of reconstruction

N.Temirgaliyev

18-24 September, 2011

COMPUTATIONAL WIDTH

$$\delta_N(\varepsilon^{(N)}) \equiv \delta_N(D_N; T, F, \varepsilon^{(N)})_Y =$$

$$= \inf_{(l^{(N)}, \varphi_N) \in D_N} \sup_{\substack{f = (f_1, ..., f_k) \in F \\ (z_1, ..., z_k) : \\ \left| l_j^{(i)}\left(f_j\right) - z_j^{(i)} \right| \le \varepsilon_j^{(i)} \\ j = 1, ..., k; i = 1, ..., N_j}} \|u(\cdot; f) - \varphi_N(z_1, ..., z_k; \cdot)\|_Y \tag{1}$$

For $\varepsilon^{(N)} = (0, ..., 0) \in R^N$ problem (1) is the problem of reconstruction under exact information.

Now, let us formulate the problem of finding the limiting error of inexact information under optimal reconstruction.

In the case $\delta_N(D_N,T,F;0)_Y \succ\prec \psi(N) \quad (N\to+\infty)$ the problem of finding the sequence $\widetilde{\varepsilon}_N = \left(\widetilde{\varepsilon}_N^{(1)},...,\widetilde{\varepsilon}_N^{(N)}\right) \; (N=1,2,...)$-limiting error of inexact information under optimal reconstruction, consists in the following: the relation $\delta_N(D_N,T,F;\widetilde{\varepsilon}_N)_Y \succ\prec \psi(N) \quad (N\to+\infty)$ holds and for any tending to $+\infty$ with N for any j sequences $\left\{\eta_N^{(j)}\right\} \quad (N=1,2,\, ...;\, j=1,2,...,N)$ the equality

$$\overline{\lim_{N\to\infty}}\frac{\delta_N\left(D_N,T,F;\left(\eta_N^{(1)}\widetilde{\varepsilon}_N^{(1)},...,\eta_N^{(N)}\widetilde{\varepsilon}_N^{(N)}\right)\right)_Y}{\delta_N(D_N,T,F;0)_Y}=$$

$$=\overline{\lim_{N\to\infty}}\frac{\delta_N\left(D_N,T,F;\left(\eta_N^{(1)}\widetilde{\varepsilon}_N^{(1)},...,\eta_N^{(N)}\widetilde{\varepsilon}_N^{(N)}\right)\right)_Y}{\psi(N)}=+\infty$$

is valid.

We start detailed presentment of problem's statement with our approach formed as "Computational width". The main idea of "Computational width"in brief is three actions executed step by step (see [1-9] and contained references)

1^0. Find $\succ\prec\ \delta_N(0)$;

2^0. Find $\{\tilde{\varepsilon}_N\}$ such that $\delta_N(0) \succ\prec \delta_N(\tilde{\varepsilon}_N)$, and

3^0. prove $\forall \eta_N \uparrow +\infty$: for any $\overline{\lim\limits_{N\to+\infty}} \frac{\delta_N(\tilde{\varepsilon}_N \eta_N)}{\delta_N(\tilde{\varepsilon}_N)} = +\infty$.

On this theme, we divided into groups by references, 1^{st} group is [1-9], 2^{nd} group is [11-15], and 3^{rd} group is [16-20]. They are distinguished under statements, use different names for the same mathematical objects, and, of course, formulated results.

A large series of works about considering subject were done by J.F. Traub, G. Wasilkowski and L. Plaskota, their co-authors and followers (see [16-20] and contained references). They investigate the minimization problem total costs of "noisy information"z_j in (0.1)-(0.2). Simultaneously, they pay big attention for relationships between different concretizations of general problem (0.1)-(0.2).

In the monograph "Noisy information and computational complexity"[16] (also see [17-20]), the reconstruction problem under inexact information was investigated in terms of minimal cost for construction of Δ-approximation ($\Delta \geq 0$).

For linear (on F) functional l and $f \in F$ let $c(\varepsilon) = c^{abc}(\varepsilon)$ and $c(\varepsilon) = c^{rel}(\varepsilon)$ be a *cost* of finding a number z such that $|z - l(f)| \leq \tau(\varepsilon; f)$, where $\tau(\varepsilon; f) = \varepsilon$ and $\tau(\varepsilon; f) = \varepsilon\,|l(f)|$ in absolute and relative cases corresponding ($\varepsilon \geq 0$).

Suppose, $c(\varepsilon)$ is independent on *l*, *f* and z and is non-decreasing function of $\varepsilon \geq 0$.

Total cost for finding information $z = (z_1, ..., z_N)$ with precision $\varepsilon^{(N)} = (\varepsilon_1, ..., \varepsilon_N)$, by definition, is

$$c(\varepsilon^{(N)}) = \sum_{j=1}^{N} c(\varepsilon_j).$$

According to [16-17], а для данного $\Delta \geq 0$ number $mc(\Delta)$ is called *minimal information* cost if

$$minc(\Delta;T;F)_Y \equiv mc(\Delta) =$$

$$= inf\{c(\varepsilon^{(N)}) : \exists N, \exists l^{(N)}, \exists \varphi_N, \exists \tau^{(N)}(\varepsilon^{(N)};f) =$$

$$= (\tau_1(\varepsilon_1;f),...,\tau_N(\varepsilon_N;f)) : \delta_N \left(l^{(N)};\varphi_N;T;F;\tau^{(N)}(\varepsilon^{(N)};f)\right)_Y \leq \Delta\}.$$

Another direction of investigation is introduced in works of V.M. Tikhomirov, G.G. Magaril-Iliyaev, K.Yu. Osipenko, and A.G.Marchuk ([11-15]), their collaborators and followers, where they find exact *solutions of the* problem.

Demonstrate one of their results (see [11]): *In the measurement error in the metric data* $\|\rho\|_{l_n^\infty} = \max_{j=1,...,n} \left|\rho_j\right|$ *to the class* $H_\infty^\omega =$

$\{f : |f(x) - f(y)| \leq \omega(|x - y|), a \leq x, y \leq b\}$, *where* $\omega(\delta)$ *- given modulus of continuity for the points* $a \leq x_1 < x_2 < ... < x_N \leq b$, *which are given in approximate values, for every* $x, a \leq x \leq b$ *the equality*

$$r_\infty(x, \varepsilon, x_1, ..., x_N) :=$$

$$= \inf_{\varphi} \sup_{\substack{f \in H^\omega_\infty \\ \max\limits_{j=1,...,N} \left|z_j - f(x_j)\right| \leq \varepsilon}} |f(x) - \varphi(x, \varepsilon, z_1, ..., z_N)| =$$

$$= \left|f(x) - g^\lambda(x, \varepsilon, x_1, ..., x_N)\right| = \varepsilon + \omega\left(\min_{i=1,...,N} |x - x_i|\right),$$

$F = H_p^r(0,1)$ - be the Nikol'skii class (the definition is given below in §2), $Y = L^\infty \equiv C[0,1]$, where $1 \leqslant p < +\infty$, $r > 1 + \frac{1}{p}$,

$$D_N^{(*)} = \Big\{(l_0(f), ..., l_N(f)) : l_j(f) - \textit{are all linear functionals}$$

$$\textit{from the linear hull} F = H_p^r(0,1) \textit{such that} \Big|l_j(1)\Big| \leqslant 1\Big\} \times \{\varphi_N\}.$$

And, at last, $\varepsilon_N \downarrow 0\,(N \uparrow +\infty)$, $\varepsilon^{(N)} = (\varepsilon_N, ..., \varepsilon_N) \in R^N$, $\Big|l_j(f) - z_j\Big| \leqslant \varepsilon_N$ $(j = 1, ..., N)$.

Under these conditions, the following theorem is valid

where $g^\lambda(x, \varepsilon, x_1, ..., x_N)$ -written explicitly as a piecewise constant function.

Theorem. *Let numbers* $1 \le p < \infty$ *and* $r > 1 + \frac{1}{p}$ *be given and* $\tilde{\varepsilon}_N = N^{-\left(r - \frac{1}{p}\right)}$. *Then*

$$\delta_N(D_N^{(*)}, Tf = f, H_p^r\left(0,1\right), 0)_{C[0,1]} \succ\prec$$

$$\succ\prec \delta_N(D_N^{(*)}, Tf = f, H_p^r\left(0,1\right), \tilde{\varepsilon}_N)_{C[0,1]} \succ\prec N^{-\left(r - \frac{1}{p}\right)},$$

For any tending to $+\infty$ *positive sequence* $\{\eta_N\}_{N=1}^{\infty}$ *the equality*

$$\overline{\lim_{N\to\infty}} \frac{\delta_N\left(D_N^{(*)}; Tf = f, H_p^r(0,1); \tilde{\varepsilon}_N \eta_N\right)_{C[0,1]}}{\delta_N\left(D_N^{(*)}; Tf = f, H_p^r(0,1); 0\right)_{C[0,1]}} = +\infty.$$

takes place.

$F = W_p^r(0,1)$ is Sobolev class (a definition gives below in 2), where integer $r \geq 1$, $1 \leq p < q \leq +\infty$.

$D_N^{(*)} = \Big\{(l_1(f), ..., l_N(f)) : l_j(f)-$ *are all possible linear functionals on linear span over*$F = W_p^r(0,1)$.*Such that* $\Big| l_j(1) \Big| \leq 1 \Big\} \times \{\varphi_N\}$.

The main aim of our paper in the modeling case to demonstrate a pithiness of the problem $1^0 - 3^0$ of limiting error of inexact information under optimal reconstruction.

Theorem. *Let numbers* $1 \leq p < q \leq \infty$ *и* r $(r = 1, 2, 3, ...)$, *Let* $\tilde{\varepsilon}_N = N^{-\left(r - \left(\frac{1}{p} - \frac{1}{q}\right)\right)}$. *Then*

$$\delta_N(D_N^{(*)}; W_p^r\,(0,1)\,; 0)_{L^q} \succ\prec$$

$$\succ\prec \delta_N(D_N^{(*)}; W_p^r\,(0,1)\,; \tilde{\varepsilon}_N)_{L^q} \succ\prec N^{-\left(r - \left(\frac{1}{p} - \frac{1}{q}\right)\right)},$$

For any tending toк $+\infty$ *positive sequence* $\{\eta_N\}_{N=1}^{\infty}$ *the equality*

$$\overline{\lim_{N\to\infty}} \frac{\delta_N \left(D_N^{(*)}; W_p^r(0,1); \tilde{\varepsilon}_N \eta_N\right)_{L^q}}{\delta_N \left(D_N^{(*)}; W_p^r(0,1); 0\right)_{L^q}} = +\infty.$$

takes place.