Classical operators of Harmonic Analysis in variable exponent Morrey type spaces

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This talk is based on the papers:

S.Samko, 1998
S.Samko, 2011
A.Almeida, J.Hasanov and S.Samko, 2008
V.Guliyev, J. Hasanov and S.Samko, 2010
V.Guliyev, J. Hasanov and S.Samko, 2011

V E = Variable Exponent

1. **Introduction**

2. **Basics and tools from VE Lebesgue spaces**

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5. **Max., Sing. and Poten. operators in VE generalized Morrey spaces**

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1 Introduction

GENERALIZED LEBESGUE SPACES $L^{p(x)}(\Omega)$
or
VE LEBESGUE SPACES
or
FUNCTION SPACES
WITH NON-STANDARD GROWTH.

$$\int_{\Omega} |f(x)|^{p(x)}\, dx < \infty.$$

$\Longrightarrow$ **Rapid development of Harmonic Analysis:**
CLASSICAL OPERATORS :
Calderon-Zygmund singular operators
Maximal Functions
Potential operators; Sobolev type theorems

The recently appeared book:
L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, 2011.

APPLICATIONS to:

1) Variational Problems,

2) Mathematical problems of mechanics of the continuum medium and informatics.

3) Operator Theory : Spectra and Fredholmness of singular integral equations and PDE in VES.

One of the reasons of an interest to VES is a possibility to localize restrictions on the exponent $p(x)$, i.e. to impose these or those conditions on p only at those points where it is necessary.

For instance, the maximal operator in the L^p-space with the weight $|x-x_0|^\alpha$:

$$-\frac{n}{p} < \alpha < \frac{n}{p'} \quad \Longrightarrow \quad -\frac{n}{p(x_0)} < \alpha < \frac{n}{p'(x_0)}.$$

Such a localization is especially important, for instance, in the problems of spectra of PDO

The main goal in this talk are the mapping properties in VES of the following classical operators:

1) the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy,$$

2) Calderon-Zygmund type singular operator

$$Tf(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:|y-x|>\varepsilon} K(x,y)f(y)dy \qquad (1.1)$$

with the so called "standard" singular kernel, and

3) potential type operators

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\Omega} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

of variable order $\alpha(x)$.

2 Basics and tools from VE Lebesgue spaces

Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

The variable exponent $p(\cdot)$ defined on Ω is supposed to satisfy the conditions

$$1 < p_- \leq p(t) \leq p_+ < \infty, \quad t \in \Omega \tag{2.1}$$

and

$$|p(t) - p(\tau)| \leq \frac{A}{\ln \frac{1}{|t-\tau|}}, \quad |t-\tau| \leq \frac{1}{2}, \quad t, \tau \in \Omega. \tag{2.2}$$

The space $L^{p(\cdot)}(\Omega)$ is defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{2.3}$$

The Hölder inequality holds in the form

$$\int\limits_{\Omega} |f(t)g(t)|\,dx \leq k\|f\|_{p(\cdot)} \cdot \|g\|_{p'(\cdot)} \tag{2.4}$$

with $k = \frac{1}{p_-} + \frac{1}{p'_-} \leq 2$.

3 VE versions of classical Morrey spaces

3.1 Recalling the constant exponent case

The spaces which bear the name of Morrey spaces were introduced by Charles Morrey, 1938, in relation to regularity problems of solutions to PDE.

Let $\Omega \subseteq \mathbb{R}^n$ **be an open set and**

$$\widetilde{B}(x,r) = B(x,r) \cap \Omega, \qquad x \in \Omega, \quad r > 0.$$

The *Morrey space*

$$L^{p,\lambda}(\Omega), \qquad 1 \le p < \infty, \quad 0 \le \lambda \le n$$

is the Banach space of functions with the finite norm

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x\in\Omega; r>0} \left(\frac{1}{r^\lambda} \int\limits_{\widetilde{B}(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}. \qquad (3.1)$$

The space $L^{p,\lambda}(\Omega)$ **is trivial when** $\lambda > n$ **and**

$$L^{p,0}(\Omega) \cong L^p(\Omega) \quad \text{and} \quad L^{p,n}(\Omega) \cong L^\infty(\Omega).$$

In the case $\lambda \in (0,n]$**, the space** $L^{p,\lambda}(\Omega)$ **is non-separable.**

For these spaces another notation $M^{p,q}$ **is also used:**

$$\|f\|_{M^{p,q}(\Omega)} := \sup_{x\in\Omega; r>0} r^{\frac{n}{q}-\frac{n}{p}} \|f\|_{L^p(\widetilde{B}(x,r))}, \qquad 1 \le p \le q \le \infty.$$

Sometimes such Morrey spaces are called

global Morrey spaces

in contrast to *local* Morrey spaces
which are defined by the norm

$$\|f\|_{L^{p,\lambda}_{\{x_0\}}(\Omega)} := \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{\widetilde{B}(x_0,r)} |f(y)|^p dy \right)^{\frac{1}{p}}, \quad \text{where} \quad x_0 \in \Omega. \tag{3.2}$$

This is a subspace of $L^p(\Omega)$ with better "Morrey-type" behaviour just at one point x_0.

Known classical results on mapping properties:
Singular integrals:
J.Peetre, 1966;
more general: F. Chiarenza and M. Frasca, 1987.

Maximal operator:
F. Chiarenza and M. Frasca, 1987

Riesz potential operator:
S.Spanne, 1969:
$L^{p,\lambda} \to L^{q,\mu}$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\mu}{q} = \frac{\lambda}{p}$,
D.Adams, 1975:
$L^{p,\lambda} \to L^{q,\lambda}$ with the better exponent $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

3.2 Definition of the VE Morrey space: the classical case

Now we pass to the variable exponent Morrey spaces

$$L^{p(\cdot),\lambda(\cdot)}(\Omega)$$

over an open set $\Omega \subset \mathbb{R}^n$.

Let $\lambda(\cdot)$ be a measurable function on Ω with values in $[0, n]$. We define the VE Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ as the set of all integrable functions f on Ω such that

$$I_{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega,\, r>0} r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy < \infty. \tag{3.3}$$

The norm in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ may be introduced in two forms,

$$\|f\|_1 = \inf\left\{\eta > 0 : \ I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\}$$

and

$$\|f\|_2 = \sup_{x\in\Omega,\, r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)},$$

which coincide indeed:

Lemma 3.1. *For every* $f \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$

$$\|f\|_2 = \|f\|_1.$$

By the coincidence of norms we put

$$\|f\|_{p(\cdot),\lambda(\cdot)} := \|f\|_1 = \|f\|_2.$$

This definition recovers the classical Morrey space, when

$$p(x) \equiv p \quad \text{and} \quad \lambda(x) \equiv \lambda \quad \text{are constant}.$$

Furthermore, if $\lambda(x) \equiv 0$, then $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ coincides with the VE Lebesgue space $L^{p(\cdot)}(\Omega)$.

Lemma 3.2 below provides another equivalent norm in the space

$$L^{p(\cdot),\lambda(\cdot)}(\Omega),$$

when $|\Omega| < \infty$. Basically, it states that in case $\lambda(\cdot)$ is log-continuous, there is no difference in taking the parameter λ depending on x or y.

Lemma 3.2. *If Ω is bounded and $\lambda(\cdot)$ is $\log$-Hölder continuous, then the functional*

$$\|f\|_3 := \sup_{x\in\Omega,\, r>0} \left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)} \tag{3.4}$$

defines an equivalent norm in $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Lemma 3.2 is an immediate consequence of the following simple fact.

Let Ω be bounded and $\lambda(\cdot)$ satisfy the log-Hölder condition

$$|\lambda(x) - \lambda(y)| \leq \frac{A_\lambda}{-\ln|x-y|} \quad \textit{for} \quad |x-y| \leq \frac{1}{2}. \tag{3.5}$$

Then

$$\frac{1}{C} r^{-\lambda(y)} \leq r^{-\lambda(x)} \leq C r^{-\lambda(y)} \tag{3.6}$$

for all $|x-y| \leq r$.

3.3 Embeddings of variable Morrey spaces

Theorem 3.3. *Let Ω be bounded,*

$$0 \leq \lambda(x) \leq n \quad \textit{and} \quad 0 \leq \mu(x) \leq n.$$

If $p(\cdot)$ and $q(\cdot)$ are $\log$-Hölder continuous, $p(x) \leq q(x)$ and

$$\frac{n-\lambda(x)}{p(x)} \geq \frac{n-\mu(x)}{q(x)} \tag{3.7}$$

then

$$L^{q(\cdot),\mu(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(\Omega). \qquad (3.8)$$

Proof. **Let**

$$\|f\|_{q(\cdot),\mu(\cdot)} \leq 1.$$

By the definition of the norm

$$\|f\| = \inf\left\{\eta > 0: \ I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

where

$$I_{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)}dy < \infty, \qquad (3.9)$$

this is equivalent to assume that

$$I_{q(\cdot),\mu(\cdot)}(f) \leq 1.$$

We have to show that

$$I_{p(\cdot),\lambda(\cdot)}(f) \leq C$$

for some $C > 0$ **not depending on** f**. Let** $x \in \Omega$ **and** $r \in (0, d_\Omega)$**. Applying the Hölder inequality with the exponent**

$$p_1(x) = \frac{q(x)}{p(x)},$$

we get

$$r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)}dy \leq 2\, r^{-\lambda(x)} \left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)} \left\|\chi_{\widetilde{B}(x,r)}\right\|_{p_1'(\cdot)}. \qquad (3.10)$$

The estimation of norms of characteristic functions of balls in VES is known:

$$\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p_1'(\cdot)} \le C\, r^{n\left(1-\frac{p(x)}{q(x)}\right)}. \tag{3.11}$$

We have to estimate the norm

$$\left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)} = \tag{3.12}$$

$$\inf\left\{\eta>0: \int\limits_{\widetilde{B}(x,r)} |f(y)|^{q(y)}\,\eta^{-\frac{q(y)}{p(y)}}dy \le 1\right\}.$$

To this end, we use the inequality

$$\frac{1}{A}\,r^{\frac{\mu(x)}{q(y)}} \le r^{\frac{\mu(x)p(x)}{q(x)p(y)}} \le A\,r^{\frac{\mu(x)}{q(y)}} \tag{3.13}$$

easily obtained by the log-condition for $p(x)$ **and** $q(x)$**. The estimate**

$$\left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)} \le A^{p_+}\,r^{\mu(x)\frac{p(x)}{q(x)}} \tag{3.14}$$

holds, where A **is the constant from the above inequality. Indeed, by (3.13),**

$$\int\limits_{\widetilde{B}(x,r)} \left(|f(y)|\left[A^{p_+}\,r^{\mu(x)\frac{p(x)}{q(x)}}\right]^{-\frac{1}{p(y)}}\right)^{q(y)} dy$$

$$\le \int\limits_{\widetilde{B}(x,r)} \left(A^{-1}\,|f(y)|\,r^{-\frac{\mu(x)p(x)}{q(x)p(y)}}\right)^{q(y)} dy$$

$$\le r^{-\mu(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{q(y)}\,dy \le 1$$

which proves (3.14). Making use of estimates (3.11) and (3.14) in (3.10), we get

$$\int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy \le C\, r^{n-\lambda(x)-\frac{p(x)}{q(x)}[n-\mu(x)]},$$

which is dominated by $r^{\mu(x)}$ under condition (3.7). Then

$$I_{p(\cdot),\lambda(\cdot)}(f) \le c \implies \|f\|_{p(\cdot),\lambda(\cdot)} \le C$$

which proves embedding (3.8). □

4 The maximal operator in VE Morrey spaces

The following theorem in the case of constant exponents $p(x) \equiv p$ and $\lambda(x) \equiv \lambda$ was proved by F. Chianza and M. Frasca, 1987.

We put

$$\lambda_+ := \operatorname*{ess\,sup}_{x\in\Omega} \lambda(x).$$

In the sequel we suppose that

$$0 \le \lambda(x) \le \lambda_+ < n, \quad x \in \Omega. \tag{4.15}$$

Theorem 4.4. *Let Ω be a bounded open set in $\mathbb{R}^n$. Under the log-condition on $p(x)$ and condition (4.15) on $\lambda(x)$, the maximal operator M is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Proof. We have to show that

$$I_{p(\cdot),\lambda(\cdot)}(Mf) \le C \tag{4.16}$$

for all f with

$$\|f\|_{p(\cdot),\lambda(\cdot)} \leq c,$$

where $c > 0$ and $C = C(c)$ does not depend on f. We have

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\, dy = \int\limits_{\mathbb{R}^n} \left((Mf(y))^{\frac{p(y)}{p_-}} \right)^{p_-} \chi_{\widetilde{B}(x,r)}(y)dy.$$

Since Ω is bounded, we have

$$\|f\|_p \leq c\|f\|_{p,\lambda} \leq c,$$

which enables us to make use of Diening's pointwise estimate

$$(Mf(x))^{\frac{p(x)}{p_-}} \leq C \left[M \left(|f(\cdot)|^{\frac{p(\cdot)}{p_-}} \right)(x) + 1 \right]. \tag{4.17}$$

Therefore,

$$\begin{aligned} &\int\limits_{\widetilde{B}(x,r)} [Mf(y)]^{p(y)} dy \\ \leq C &\int\limits_{\Omega} \left[M \left(|f(\cdot)|^{\frac{p(\cdot)}{p_-}} \right) \right]^{p_-} \chi_{\widetilde{B}(x,r)}(y)\, dy \\ &+ C \int\limits_{\Omega} \chi_{\widetilde{B}(x,r)}(y)\, dy. \end{aligned}$$

By the Fefferman-Stein inequality (for constant p)

$$\int\limits_{\mathbb{R}^n} (Mg)(y)^p\, h(y)\, dy \leq \int\limits_{\mathbb{R}^n} g(y)^p\, (Mh)(y)\, dy,$$

valid for all non-negative functions g, h, we get

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\,dy$$

$$\le C\int\limits_{\Omega} |f(y)|^{p(y)} M\chi_{\widetilde{B}(x,r)}(y)\,dy \;+\; Cr^n.$$

By the well known estimate

$$M\chi_{B(x,r)}(y) \le \frac{Cr^n}{(|x-y|+r)^n}, \quad x,y\in\mathbb{R}^n$$

we get

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\,dy \le C \int\limits_{\widetilde{B}(x,2r)} |f(y)|^{p(y)}dy$$

$$+C\sum_{j=1}^{\infty}\int\limits_{\widetilde{B}(x,2^{j+1}r)\setminus\widetilde{B}(x,2^j r)} \frac{r^n\,|f(y)|^{p(y)}}{(|x-y|+r)^n}\,dy$$

$$+\;Cr^n \le C\left(r^{\lambda(x)}+\sum_{j=1}^{\infty}\frac{(2^{j+1}r)^{\lambda(x)}}{(2^j+1)^n}+r^n\right)$$

$$\le C\left(r^{\lambda(x)}+Cr^n\right) \le C\,r^{\lambda(x)},$$

which proves the uniform estimate

$$I_{p(\cdot),\lambda(\cdot)}(f) \le C$$

and completes the proof. □

5 Potential operators in variable Morrey spaces

It is natural to consider the potential operators

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\Omega} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

of variable order $\alpha(x)$ as well. We assume that $\alpha(\cdot)$ also satisfies the log-condition

$$|\alpha(x)-\alpha(y)| \leq \frac{C}{-\ln|x-y|} \qquad \textbf{for} \qquad |x-y| \leq \frac{1}{2}. \tag{5.1}$$

The next theorem in the case of constant p and α was proved by D. Adams, 1975.

Theorem 5.1. ***Let Ω be bounded. Let $p(x)$ satisfy the log-condition and the condition***

$$1 < p_- \leq p(x) \leq p_+ < \infty$$

and $\alpha(x)$ fulfill the log-condition and the conditions

$$\inf_{x\in\Omega} \alpha(x) > 0, \qquad \sup_{x\in\Omega} [\lambda(x)+\alpha(x)p(x)] < n, . \tag{5.2}$$

Then

$$I^{\alpha(\cdot)}: \ L^{p(\cdot),\lambda(\cdot)}(\Omega) \to L^{q(\cdot),\lambda(\cdot)}(\Omega),$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}.$$

Proof. **Let $\|f\|_{p(\cdot),\lambda(\cdot)} \leq 1$. We use the standard decomposition (Hedberg's trick):**

$$I^{\alpha(\cdot)} f(x) = \left(\int\limits_{B(x,2r)} + \int\limits_{\mathbb{R}^n \setminus B(x,2r)} \right) \frac{f(y)\, dy}{|x-y|^{n-\alpha(x)}} \tag{5.3}$$

$$=: F(x,r) + G(x,r),$$

where $f(y)$ was continued by zero beyond Ω. The pointwise estimate

$$|F(x,r)| \leq C\, r^{\alpha(x)} M f(x) \tag{5.4}$$

with a constant $C > 0$ not depending on f and x is well known in the case of constant α and is also valid for variable $\alpha(\cdot)$, under the condition

$$\inf_{x \in \Omega} \alpha(x) > 0$$

(see [Samko 1998]. For $G(x,r)$ we have

$$|G(x,r)| \leq$$

$$C \sum_{j=1}^{\infty} \int\limits_{B(x,2^{j+1}r) \setminus B(x,2^j r)} \frac{|f(y)| (2^j r)^{-\frac{\lambda(x)}{p(y)}} dy}{|x-y|^{n-\alpha(x)-\frac{\lambda(x)}{p(y)}}}.$$

Since $p(\cdot)$ satisfies the log-condition, we then also have

$$|G(x,r)| \leq$$

$$C \sum_{j=1}^{\infty} \int\limits_{B(x,2^{j+1}r) \setminus B(x,2^j r)} \frac{|f(y)|\, (2^j r)^{-\frac{\lambda(x)}{p(y)}} dy}{|x-y|^{n-\alpha(x)-\frac{\lambda(x)}{p(x)}}}.$$

Applying the Hölder inequality, we get

$$|G(x,r)| \leq \tag{5.5}$$

$$C\sum_{j=1}^{\infty}\left\| |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}} \right\|_{p'(\cdot),\mathbb{R}^n\setminus B(x,2^jr)}$$
$$\times \left\| (2^jr)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)}.$$

The factor

$$\left\| (2^jr)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)}$$

is uniformly bounded. To see this, it suffices to show that the modularis bounded:

$$I_{p(\cdot)}\left((2^jr)^{-\frac{\lambda(x)}{p(\cdot)}} f\chi_{(B(x,2^{j+1}r)} \right) \leq C < \infty$$

which is valid, since

$$I_{p(\cdot)}\left((2^jr)^{-\frac{\lambda(x)}{p(\cdot)}} f\chi_{(B(x,2^{j+1}r)} \right) \leq$$
$$(2^{j+1}r)^{-\lambda(x)} \int\limits_{B(x,2^{j+1}r)} |f(y)|^{p(y)}dy \leq C < \infty$$

by the definition of the variable Morrey-norm. Therefore,

$$|G(x,r)| \leq C\sum_{j=1}^{\infty}\left\| |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}} \right\|_{p'(\cdot),\mathbb{R}^n\setminus B(x,2^jr)}. \quad (5.6)$$

Now we use the following estimate (Samko, 1998):

Let $\sup_{x\in\Omega}\beta(x) < \infty, \quad \inf_{x\in\Omega}\beta(x)p(x) > n.$

Then $\left\| \frac{\chi_{\mathbb{R}^n\setminus B(x,r)}(\cdot)}{|x-\cdot|^{\beta(x)}} \right\|_{p(\cdot)} \leq C.r^{\frac{n}{p(x)}-\beta(x)}$

which transforms (5.6) **to**

$$|G(x,r)| \leq \tag{5.7}$$

$$C_1 \sum_{j=1}^{\infty} (2^j r)^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}} \leq C_2\, r^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}},$$

where the series

$$C_2 = C_1 \sum_{j=1}^{\infty} 2^{-aj},$$

$$a = \frac{1}{p_+} \inf_{x \in \Omega} [n - \lambda(x) - \alpha(x) p(x)] > 0,$$

defining the constant C_2**, is convergent.**

Thus, from (5.4) **and** (5.7) **we have**

$$\left| I^{\alpha(\cdot)} f(x) \right| \leq C\, r^{\alpha(x)} M f(x) + C_2\, r^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}}.$$

Minimizing at $r = (M f(x))^{-\frac{p(x)}{n-\lambda(x)}}$ **we get**

$$\left| I^{\alpha(\cdot)} f(x) \right| \leq c\, (M f(x))^{\frac{p(x)}{q(x)}}.$$

Hence, by the boundedness of the maximal operator in variable $L^{p(\cdot)}$**-spaces, we have**

$$\int_{\widetilde{B}(x,r)} \left| I^{\alpha(\cdot)} f(y) \right|^{q(y)} dy \leq$$

$$c \int_{\widetilde{B}(x,r)} (M f(y))^{p(y)}\, dy \leq c\, r^{\lambda(x)},$$

which completes the proof of the theorem. □

Corollary 5.2. ***Under the assumptions of Theorem 5.1 the fractional maximal operator $M^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\lambda(\cdot)}(\Omega)$.***

This statement in the case of constant exponents p, λ and α was proved by M.Fazio and M.A.Raguza, 1991.

The proof follows from Theorem 5.1 in view of the pointwise estimate

$$M^{\alpha(\cdot)}f(x) \le c\, I^{\alpha(\cdot)}(|f|)(x), \quad 0 < \alpha(x) < n, \tag{5.8}$$

where $c > 0$ does not depend on f and x.

In the case of constant p, λ and α there is known also the norm equivalence of $I^\alpha f$ and $M^\alpha f$ in Morrey spaces (Adams, Xiao, 2004).

This inequality, well known for constant α, is also valid for variable $\alpha(x)$ with

$$c = \sup_{x\in\Omega} \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\frac{\alpha(x)}{n}} < \infty.$$

6 Potential operators: the limiting case $p(x) \equiv \frac{n-\lambda(x)}{\alpha(x)}$

As is known, in the limiting case, the mapping properties of the Riesz potential operator $I^{\alpha(\cdot)}$ and the fractional maximal operator $M^{\alpha(\cdot)}$ are different, the

range of the former is in L^∞, of the latter in the space

$$BMO = \{f : M^\sharp f \in L^\infty\}.$$

Theorem 6.3. ***Let Ω be bounded, $p(x)$ satisfy the log-condition and the condition***

$$1 < p_- \leq (x) \leq p_+ < \infty,$$

let $\inf\limits_{x\in\Omega} \alpha(x) > 0$ and

$$p(x) = \frac{n - \lambda(x)}{\alpha(x)}.$$

The operator $M^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^\infty(\Omega)$:

$$\|M^{\alpha(\cdot)} f\|_\infty \leq C\, \|f\|_{p(\cdot),\lambda(\cdot)}. \tag{6.9}$$

The Riesz potential operator I^α:
the $L^{p,\lambda} \to BMO$-boundedness for constant p, λ and α is known: S.Spanne, 1969.

It proves to be also valid for variable $p(\cdot)$ and $\lambda(\cdot)$ at the least when α is constant:

Theorem 6.4. ***Let***

$$\lambda(x) \geq 0, \;\; 0 < \alpha < n, \;\; \sup_{x\in\Omega} \lambda(x) < n - \alpha,$$

let $p(x)$ satisfy the log-condition and let

$$p(x) = \frac{n - \lambda(x)}{\alpha}.$$

Then

$$I^\alpha : \; L^{p(\cdot),\lambda(\cdot)}(\Omega) \to \; BMO(\Omega).$$

Proof. **Use the pointwise estimate (Adams, 1975):**

$$M^\sharp(I^\alpha f)(x) \le c\, M^\alpha f(x), \qquad x \in \Omega. \tag{6.10}$$

□

7 VE generalized Morrey spaces

Everywhere in the sequel we suppose that:

$\Omega \subset \mathbb{R}^n$ **is an open bounded set,** $1 < p_- \le p(x) \le p_+ < \infty$ **and** $p(\cdot)$ **is log-continuous.**

We define now the VE generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ **by the norm**

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

We recover the previous space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ **under the choice**

$$\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}.$$

Everywhere in the sequel we assume that

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0 \tag{7.1}$$

which makes the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ **nontrivial.**

Known facts in the case of constant p (**V.Guliyev, T.Mizuhara, E.Nakai**):

Theorem 7.1. ***(V.Guliyev, 1994) Let*** $1 < p < \infty$ ***and***

$$\int_r^\infty \frac{\omega(x,t)}{t}\,dt \le c_1\,\omega(x,r).$$

Then the operators M ***and*** T ***are bounded in the space*** $\mathcal{M}^{p,\omega(\cdot)}(\mathbb{R}^n)$.

Theorem 7.2. ***(V.Guliyev, 1994) Let***

$$0 < \alpha < n, \quad 1 < p < \frac{n}{\alpha} \quad \textit{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

and

$$\int_r^\infty t^\alpha \omega_1(x,t)\frac{dt}{t} \le c_1\,r^\alpha \omega_2(x,r).$$

Then the operators M^α ***and*** I^α ***are bounded from*** $\mathcal{M}^{p,\omega_1(\cdot)}(\mathbb{R}^n)$ ***to*** $\mathcal{M}^{q,\omega_2(\cdot)}(\mathbb{R}^n)$.

But Adams result was not known

7.1 The maximal operator M in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$

The main task: to estimate means over balls of the image an operator directly via means of the function itself.

Theorem 7.3. ***The estimate***

$$\|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{p(x)}} \int\limits_t^\ell \frac{\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}{r^{\frac{n}{p(x)}-1}} dr, \qquad 0 < t < \frac{\ell}{2}$$

is valid.

Proof. **The main points:**

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\backslash\widetilde{B}(x,2t)}(y), \tag{7.2}$$

For Mf_2 we prove:

$$\int\limits_{\Omega\backslash\widetilde{B}(x,t)} \frac{|f(y)|}{|x-y|^n} dy \le C \int\limits_t^\ell s^{-\frac{n}{p(x)}-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} ds. \tag{7.3}$$

To this end, we choose $\beta > \frac{n}{p_-}$ and proceed as follows

$$\int\limits_{\Omega\backslash\widetilde{B}(x,t)} \frac{|f(y)|dy}{|x-y|^n} \le \beta \int\limits_{\Omega\backslash\widetilde{B}(x,t)} \frac{|f(y)|}{|x-y|^{n-\beta}} \left(\int\limits_{|x-y|}^\ell s^{-\beta-1} ds \right) dy$$

$$= \beta \int\limits_t^\ell s^{-\beta-1} \left(\int\limits_{\{y\in\Omega: 2t \le |x-y| \le s\}} |x-y|^{-n+\beta} |f(y)| dy \right) ds$$

$$\le C \int\limits_t^\ell s^{-\beta-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \||x-y|^{-n+\beta}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))} ds.$$

We then make use of the estimate (S.Samko, 1998):

$$\||x-y|^{\nu(x)}\chi_{B(x,r)}(y)\|_{p(y)} \le Cr^{\nu(x)+\frac{n}{p(x)}}, \qquad x \in \Omega, 0 < r < \ell = \text{diam } \Omega,$$

where $\sup \nu(x) < \infty$ and $\inf[n + \nu(x)p(x)] > 0$ and arrive at (7.3). $\square$

Theorem 7.4. ***Let***

$$\int_r^\ell \frac{\omega(x,t)}{t}\,dt \le C\,\omega(x,r). \tag{7.4}$$

Then the maximal operator M is bounded in the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$.

By the same method, via the estimate of the means:

$$\|Tf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{p(x)}} \int_t^\ell \frac{\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}{r^{\frac{n}{p(x)}+1}}dr, \quad 0 < t \le \frac{\ell}{2},$$

(with a proof modified), the singular operators may be treated:

Theorem 7.5. ***Let $\omega(x,t)$ fulfill condition (11.6). Then the singular integral operator T is bounded in the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$.***

7.2 Riesz potential operator in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$

For the potential operators we give two versions:
Spanne-type result
and

Adams-type result. This is new even in the case of constant p.

Recall that:

Spanne's result: $\alpha p < n$ **and** $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$
Adams' result : $\alpha p < n - \lambda$ **but** $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

7.2.1 Spanne type result

We assume that

$$\inf_{x\in\Omega \alpha(x)>0} \sup_{x\in\Omega} \alpha(x)p(x) < n \quad \textbf{and} \quad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}.$$

We exploit the same idea of estimation of means over balls:

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\widetilde{B}(x,t))} \leq C t^{\frac{n}{q(x)}} \int_t^l r^{-\frac{n}{q(x)}-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} dr, \quad 0 < t \leq \frac{\ell}{2},$$

and obtain the following statement:

Theorem 7.6. ***Let***

$$\int_r^\ell t^{\alpha(x)} \omega_1(x,t) \frac{dt}{t} \leq C\, \omega_2(x,r). \tag{7.5}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

7.2.2 Adams type result

The approach is different.

First:

Theorem 7.7. ***The pointwise estimate***

$$|I^{\alpha(\cdot)}f(x)| \le Ct^{\alpha(x)}\,Mf(x) + C\int\limits_t^{\ell} \frac{\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}{r^{1+\frac{n}{p(x)}-\alpha(x)}}dr \qquad 0 < t \le \frac{\ell}{2}, \tag{7.6}$$

is valid.

Theorem 7.8. ***Let*** $q(x) > p(x)$ ***on*** Ω ***and***

$$\int\limits_r^{\ell} \omega(x,t)\frac{dt}{t} \le C\,\omega(x,r),$$

$$\int\limits_r^{\ell} t^{\alpha(x)-1}\,\omega(x,t)dt \le Cr^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}. \tag{7.7}$$

Then the operator $I^{\alpha(\cdot)}$ ***is bounded from*** $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ ***to*** $\mathcal{M}^{q(\cdot),\omega^{\frac{p(\cdot)}{q(\cdot)}}}(\Omega)$***.***

Proof. **By the above pointwise estimate and the condition** (8.16)**, we obtain**

$$|I^{\alpha(\cdot)}f(x)| \le Cr^{\alpha(x)}\,Mf(x) + Cr^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}\,\|f\|_{\mathcal{M}^{p(\cdot),\omega}(\Omega)}.$$

Minimize at $r = \left(\frac{\|f\|_{\mathcal{M}^{p(\cdot),\omega}}}{Mf(x)}\right)^{\frac{q(x)-p(x)}{\alpha(x)q(x)}}$**:**

$$|I^{\alpha(\cdot)}f(x)| \leq C(Mf(x))^{\frac{p(x)}{q(x)}} \|f\|^{1-\frac{p(x)}{q(x)}}_{\mathcal{M}^{p(\cdot),\omega}(\Omega)}.$$

and apply Diening's theorem on the VE boundedness of the maximal operator in Lebesgue spaces. □

On a generalization of the Sobolev-Adams exponent:

By the condition

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0$$

of non-triviality of the space, we may assume that

$$\omega(x,r) \geq 1.$$

For the exponent $q(x)$, from (8.16) there follows the following bound

$$\frac{1}{q(x)} \geq \frac{1}{p(x)} - \frac{\alpha(x)}{m(x)},$$

where

$$m(x) = p(x)\left[\alpha(x) - \overline{\lim_{r\to 0}} \frac{\ln \int\limits_r^\ell t^{\alpha(x)-1} w(r,t)\,dt}{\ln\, r}\right].$$

The corresponding exponent $q(x)$ given by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{m(x)}, \tag{7.8}$$

might be called the *Sobolev-Adams-type exponent* corresponding to the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$.

In particular, for the Morrey space $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$ when

$$\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}},$$

from (7.8) we recover Adams' exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}.$$

8 Another generalization of Morrey spaces

A natural generalization of the Morrey space consists in replacing

$$\sup_{r>0} -\textbf{norm}$$

by the $\|\cdot\|_{L^\theta(0,\infty)}$-norm:

$$\|f\|_{L^{p,\theta,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{\theta}{p}} \frac{dr}{r} \right)^{\frac{1}{\theta}}. \quad (8.1)$$

Such spaces first appeared in
Umea Lectures of D. Adams 1981, as an episode. There the Sobolev type theorem for the Riesz potential operator is such spaces was stated with the sketch of the proof via the capacitory approach.

Various operators of harmonic analysis in such spaces were thoroughly investigated by
V. Guliyev in 1994-1997,

and later, to a wider extent, in a series of papers of V. Burenkov, H. Guliyev, V. Guliyev and others.

What about VE versions of such spaces?

V.Guliyev and S.Samko, 2011:

Definition 8.1. We define the space

$$\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$$

by the norm

$$\|f\|_{\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} = \sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)}.$$

(The case $\theta(r) \equiv \infty$ may be also admitted).

Note:

we impose the standard log-condition on the exponent $p(x)$ but this log-condition *is not required* for the variable exponent $\theta(r)$. In some statements we even do not impose the log-type decay condition on $\theta(r)$ as $r \to 0$, but we introduce such a decay condition when we wish to obtain easy to check conditions.

In the sequel we assume that $\theta(t)$ is a bounded measurable function with values in $[1,\infty)$.

We also assume that

$$\sup_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(0,\ell)} < \infty, \tag{8.2}$$

which is sufficient for the embedding

$$L^\infty(\Omega) \hookrightarrow \mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega). \tag{8.3}$$

If Ω satisfies the condition

$$\inf_{x\in\Omega} |\Omega \cap B(x,r)| \geq cr^n, \tag{8.4}$$

then condition (8.2) is also necessary.

Proof. ***Necessity***:

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\Omega)} \geq Cr^{\frac{n}{p(x)}},$$

which is checked in terms of the modular:

$$\int_\Omega \left(\frac{\chi_{B(x,r)}(y)}{\lambda |B(x,r)|^{\frac{1}{p(x)}}} \right)^{p(y)} dy \geq 1$$

for some $\lambda > 0$. □

The condition on $\omega(x,r)$:

$$\inf_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(\delta,\ell)} > 0, \tag{8.5}$$

for some $\delta > 0$ guarantees the embedding

$$\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \tag{8.6}$$

***Notation*:**

Given $\delta \in (0,\ell)$, by $\mathcal{W}(\delta,\ell)$ we denote the set of duplets (θ,ω) satisfying condition (8.5).

Thus with $(\theta,\omega) \in \mathcal{W}(\delta,\ell)$ we always have the embedding

$$L^\infty(\Omega) \hookrightarrow \mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \tag{8.7}$$

8.1 On the maximal operator

Theorem 8.2. ***Let p be log-continuous and***

$$1 < \theta_1^- \le \theta_1(t) \le \theta_1^+ < \infty, \qquad 1 \le \theta_2^- \le \theta_2(t) \le \theta_2^+ < \infty \tag{8.8}$$

Let also there exist $\delta > 0$ such that

$$\theta_1(t) \le \theta_2(t) \quad \textit{for} \quad t \in (0, \delta), \tag{8.9}$$

and $(\theta_1, \omega_1) \in \mathcal{W}(\delta, \ell)$. If the condition

$$\sup_{x\in\Omega,\, 0<t<\delta} \int_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int_t^\delta \frac{dr}{[r\omega_1(x,r)]^{[\widetilde{\theta}_1(\xi)]'}} \right)^{\frac{\theta_2(\xi)}{[\widetilde{\theta}_1(\xi)]'}} d\xi < \infty, \tag{8.10}$$

holds, where

$$\widetilde{\theta}_1(\xi) = \inf_{s\in(\xi,\ell)} \theta_1(s),$$

then the operator M is bounded from $\mathcal{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathcal{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$.

Remark 8.3. Condition (8.10) is imposed only in an arbitrarily small neighbourhood $(0, \delta)$ and no log-condition or even a log-type decay condition at the point $r = 0$ is imposed on on $\theta_1(r)$ and $\theta_2(r)$.

Corollary 8.4. ***In the case***

$$\omega_1(x,r) = \omega_2(x,r) = r^{\beta(x)}$$

and $\theta_1(r) = \theta_2(r) =: \theta(r)$, under the assumption that

$$\inf_{x\in\Omega} \beta(x) > -\frac{1}{\inf\limits_{t\in(\delta,\ell)} \theta(t)}, \tag{8.11}$$

condition (8.10) ***takes the form***

$$\sup_{x\in\Omega,\, 0<t<\delta} \int_0^t \left(\frac{\xi}{t}\right)^{\beta(x)\theta(\xi)} \frac{d\xi}{t^{\frac{\theta(\xi)}{\widetilde{\theta}(\xi)}}} < \infty \tag{8.12}$$

(again: no log-condition on $\theta(\xi)$ ***and*** $\beta(x)$***); in particular, when*** $\theta(t) \equiv \theta =$ ***const*** $(1 < \theta < \infty)$***, the conditions***

$$p \quad \textit{is log-continuous} \quad \textit{and} \quad \inf_{x\in\Omega} \beta(x) > -\frac{1}{\theta}$$

are sufficient for the boundedness of the maximal operator M ***in the space*** $\mathcal{M}^{p(\cdot),\theta,r^{\alpha(x)}}(\Omega)$.

Let

$$\omega_1(x,r) = r^{\beta(r)}, \quad \omega_2(x,r) = r^{\gamma(r)}. \tag{8.13}$$

In this case, conditions on $\theta_1(\cdot)$ **and** $\theta_2(\cdot)$ **may be given in a very simple form, if**

$$\beta(x), \gamma(x) \quad \text{and} \quad \theta_1(r), \theta_2(r)$$

satisfy the log-type decay condition at $x = 0$ **and** $r = 0$. **Then**

Theorem 8.5. ***The maximal operator*** M ***is bounded from*** $\mathcal{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ ***to*** $\mathcal{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$, ***if***

$$-\frac{1}{\theta_1(0)} < \beta(0) \le \gamma(0) \tag{8.14}$$

and

$$\frac{1}{\theta_2(0)} = \frac{1}{\theta_1(0)} + \beta(0) - \gamma(0). \tag{8.15}$$

Similar results are also obtained for singular Calderon-Zygmund operators

8.2 On potential operators

We omit Spanne type result, but dwell on the next Adams-type result.

Theorem 8.6. ***Let*** p, α ***be log-continuous functions, let*** $q(x) > p(x)$ ***on*** Ω***, let*** θ_1 ***and*** θ_2 ***satisfy assumptions***

$$1 < \theta_1^- \le \theta_1(t) \le \theta_1^+ < \infty, \qquad 1 \le \theta_2^- \le \theta_2(t) \le \theta_2^+ < \infty$$

and let $\omega(x,r)$ ***fulfill the condition***

$$\left\| \frac{t^{\alpha(x)-1}}{\omega_1(x,t)} \right\|_{L^{\theta_1'(\cdot)}(r,\ell)} \le C r^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}. \tag{8.16}$$

If there exist a $\delta \in (0,\ell)$ ***such that***

$$(\theta_1, \omega_1) \in \mathcal{W}(\delta, \ell),$$

while on $(0,\delta)$ ***there hold conditions***

$$\theta_1(t) \le \theta_2(t), \qquad 0 < t < \delta,$$

and

$$\sup_{x\in\Omega,\, 0<t<\delta} \int_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int_t^\delta \frac{dr}{[r\omega_1(x,r)]^{[\widetilde{\theta}_1(\xi)]'}} \right)^{\frac{\theta_2(\xi)}{[\widetilde{\theta}_1(\xi)]'}} d\xi < \infty,$$

then the operators $M^{\alpha(\cdot)}$ ***and*** $I^{\alpha(\cdot)}$ ***are bounded from*** $\mathcal{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ ***to*** $\mathcal{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$***.***

Recall:

$$\mathcal{W}(\delta,\ell) = \left\{ (\theta, \omega : \inf_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(\delta,\ell)} > 0) \right\}.$$

9 "Complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$

V.Guliyev, 1994 (constant $p \in [1,\infty)$):

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\Omega\setminus B(x_0,r))},$$

where $x_0 \in \Omega$.
Particular case:

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(\mathbb{R}^n\setminus B(x_0,r))} < \infty, \quad 0 \le \lambda < n \tag{9.17}$$

In contrast to the Morrey space, where we measure the REGULARITY of a function f near a point x, the norm (9.17) is aimed to measure a "BAD" BEHAVIOUR

We consider such local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with variable exponent $p(\cdot)$.

9.1 Relations with weighted Lebesgue spaces; the case of constant p

The standard notation:

$$L^p(\Omega,\varrho) = \left\{ f : \int_\Omega \varrho(y)|f(y)|^p\,dy < \infty \right\}.$$

Theorem 9.1. ***Let*** $1 \le p < \infty$, $0 \le \lambda \le n$ ***and*** $A > diam\,\Omega$***. Then***

$$L^p(\Omega, |y-x_0|^{\lambda(p-1)}) \hookrightarrow {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} L^p\left(\Omega, \frac{|y-x_0|^{\lambda(p-1)}}{\left(\ln \frac{A}{|y-x_0|}\right)^{1+\varepsilon}}\right) \tag{9.1}$$

where both the embeddings are strict, with the counterexamples

$$f(x) = \frac{1}{|x-x_0|^{\frac{n}{p}+\frac{\lambda}{p'}}}$$

and

$$g(x) = \frac{\ln\left(\ln \frac{B}{|x-x_0|}\right)}{|x-x_0|^{\frac{n}{p}+\frac{\lambda}{p'}}}, \quad B > e^e\, diam\,\Omega,$$

for the first and second embeddings, respectively.

Remark 9.2. **Similar to the left-hand side embedding in** (9.1)**, the following embedding holds for the case of general spaces** ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$**:**

$$L^p(\Omega, \rho(|y-x_0|)) \hookrightarrow {}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$$

where

$$\inf_{r>0} \frac{\rho(r)\omega^p(r)}{r^{n(p-1)}} > 0.$$

The weak weighted space:

$$\|f\|_{wL^p(\Omega,|y-x_0|^{\lambda(p-1)})} = \sup_{t>0} t\,[\mu\{x\in\Omega : |f(x)| > t\}]^{\frac{1}{p}} < \infty.$$

where

$$\mu(E) = \int\limits_E |y - x_0|^{\lambda(p-1)} dy.$$

Theorem 9.3. ***Let*** $1 \le p < \infty$ ***and*** $0 < \lambda \le n$***. Then***

$$L^p(\Omega, |y - x_0|^{\lambda(p-1)}) \hookrightarrow {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega) \hookrightarrow wL^p(\Omega, |y - x_0|^{\lambda(p-1)}). \tag{9.2}$$

10 VE local "complementary" generalized Morrey spaces

Definition 10.1. **Let** $x_0 \in \Omega$, $1 \le p(x) \le p_+ < \infty$. **The** ***VE generalized local "complementary" Morrey space*** ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ **is defined by the norm**

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}.$$

Particular case:

$$\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda}_{\{x_0\}}(\Omega)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^{p(\cdot)}(\Omega\setminus B(x_0,r))} < \infty, \qquad 0 \le \lambda < n \tag{10.1}$$

We assume that

$$\sup_{0<r<\ell} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} < \infty, \qquad \ell = \text{diam } \Omega, \tag{10.2}$$

then the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ **non-trivial, since it contains** $L^{p(\cdot)}(\Omega)$ **in this case.**

Remark 10.2. Suppose that

$$\inf_{\delta<r<\ell} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} > 0$$

for every $\delta > 0$. **Then**

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} \sim \|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(B(x_0,\delta))} + \|f\|_{L^{p(\cdot)}(\Omega\backslash B(x_0,\delta))},$$

i.e. the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ **may be interpreted as the space of functions** ***whose restrictions onto*** $B(x_0,\delta)$ ***are in local "complementary" Morrey space*** ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(B(x_0,\delta))$ ***with the exponent*** $p(\cdot)$ ***close to the constant value*** $p(x_0)$ ***and the restrictions onto the exterior*** $\Omega\backslash B(x_0,\delta)$ ***are in the VE Lebesgue space*** $L^{p(\cdot)}$***.***

The space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ **may contain functions with a non-integrable singularity at the point** x_0**.**

To study operators in ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$, **we need its embedding into** $L^1(\Omega)$.

To this end:

Lemma 10.3. ***Let*** $f \in L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))$ ***for every*** $s \in (0,\ell)$. ***Then***

$$\int_{\widetilde{B}(x_0,t)} |f(y)|dy \le C\int_0^t \frac{\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}}{s^{1-\frac{n}{p'(x_0)}}}ds \qquad (10.3)$$

for every $t \in (0,\ell)$***.***

Proof. **We use the following trick:**

$$\int\limits_{\widetilde{B}(x_0,t)} |f(y)|dy = \beta \int\limits_{\widetilde{B}(x_0,t)} \frac{|f(y)|}{|x_0-y|^\beta} \left(\int\limits_0^{|x_0-y|} s^{\beta-1} ds \right) dy \quad (10.4)$$

the parameter $\beta > 0$ which will be chosen later

$$= \beta \int\limits_0^t s^{\beta-1} \left(\int\limits_{\{y\in\Omega: s<|x_0-y|<t\}} \frac{|f(y)|}{|x_0-y|^\beta} dy \right) ds.$$

Hence

$$\int\limits_{\widetilde{B}(x_0,t)} |f(y)|dy \le C \int\limits_0^t s^{\beta-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))} \||x_0-y|^{-\beta}\|_{L^{p'(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))} ds.$$

We then make use of the VE-estimate (S.Samko, 1998):

$$\||x-y|^{\nu(x)}\chi_{B(x,r)}(y)\|_{p(y)} \le Cr^{\nu(x)+\frac{n}{p(x)}}, \qquad x\in\Omega, 0<r<\ell = \mathrm{diam}\ \Omega,$$

where $\sup \nu(x) < \infty$ **and** $\inf[n+\nu(x)p(x)] > 0$**. This is possible if we choose** $\beta > \frac{n}{p'_-}$ **and then arrive at** (10.3). □

Corollary 10.4. ***The following embeddings hold***

$$L^{p(\cdot)}(\Omega) \hookrightarrow {}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega) \hookrightarrow L^1(\Omega) \qquad (10.5)$$

where the 1st embedding is guaranteed by the condition

$$\sup_{0<r<\ell} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} < \infty, \qquad \ell = \ \mathrm{diam}\ \ \Omega,$$

and the 2nd one by the condition

$$\int_0^\ell \frac{\omega(r)\,dr}{r} < \infty. \tag{10.6}$$

11 The maximal operator in the spaces $\overset{\circ}{\mathcal{M}}{}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$

Estimation of means over exteriors of balls:

Theorem 11.1. ***Let*** $f \in L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))$ ***for every*** $r \in (0,\ell)$ ***and let the integral***

$$\int_0^\ell r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))}dr \tag{11.1}$$

converge. Then

$$\|Mf\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le \frac{C}{t^{\frac{n}{p'(x_0)}}}\int_0^t r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))}dr \tag{11.2}$$

for every $t \in (0,\ell)$.

Proof. **We represent** f **as**

$$f = f_1+f_2, \qquad f_1(y) = f(y)\chi_{\Omega\backslash\widetilde{B}(x_0,t)}(y) \qquad f_2(y) = f(y)\chi_{\widetilde{B}(x_0,t)}(y). \tag{11.3}$$

1^o. ***Estimation of*** Mf_1 ***is easier*****:**
direct application of VE-theorem of L.Diening for the

maximal operator:

$$\|Mf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le \|Mf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))}.$$

By the monotonicity of the norm we have

$$\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le \frac{C}{t^{\frac{n}{p'(x_0)}}} \int_0^t r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr \quad (11.4)$$

and then

$$\|Mf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le \frac{C}{t^{\frac{n}{p'(x_0)}}} \int_0^t r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr.$$

2^o. ***Estimation of*** Mf_2***.*** **This case needs the application of the above estimate** (10.3)**.**

For $z \in \Omega\backslash\widetilde{B}(x_0,2t)$ **we have**

$$\begin{aligned} Mf_2(z) &= \sup_{r>0} \frac{1}{|B(z,r)|} \int_{\widetilde{B}(z,r)} |f_2(y)|dy \\ &\le \sup_{r\ge t} \int_{\widetilde{B}(x_0,t)\cap B(z,r)} \frac{|f(y)|}{|y-z|^n} dy \\ &\le \frac{C}{|x_0-z|^n} \int_{\widetilde{B}(x_0,t)} |f(y)|dy. \end{aligned}$$

Then by (10.3)

$$Mf_2(z) \le \frac{C}{|x_0-z|^n} \int_0^t s^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))} ds. \quad (11.5)$$

Therefore,

$$\|Mf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \leq$$

$$C\int\limits_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds\ \left\||x_0-z|^{-n}\right\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))}.$$

It remains to apply the already familiar VE-estimate to the last norm:

$$\|Mf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \leq Ct^{-\frac{n}{p'(x_0)}}\int\limits_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds.$$

□

Theorem 11.2 for the complementary Morrey spaces is, in a sense, a counterpart to Theorem 11.2 we gave in the first part for the usual Morrey spaces.

Theorem 11.2. ***Let***

$$\int\limits_0^r \frac{\omega(t)}{t}\,dt \leq C\,\omega(r). \tag{11.6}$$

Then the maximal operator M ***is bounded in the space*** $^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$.

Recall:

$$\int\limits_r^\ell \frac{\omega(x,t)}{t}\,dt \leq C\,\omega(x,r) \quad \textbf{for usuall Morrey spaces.}$$

Proof. **We have**

$$\|Mf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} = \sup_{t\in(0,\ell)} \frac{t^{\frac{n}{p'(x_0)}}}{\omega(t)} \|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}.$$

Apply the previous Theorem 11.1 to the norm $\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}$**:**

$$\|Mf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} \le C \sup_{0<r\le\ell} \omega^{-1}(r) \int_0^r r^{-\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} dt.$$

Hence

$$\begin{aligned}\|Mf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} \sup_{r\in(0,\ell)} \frac{1}{\omega(r)} \int_0^r \omega(r)\frac{dt}{t} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)}.\end{aligned}$$

□

Singular integral operator; similar approach:

$$\|Tf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^{2t} r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))} dr,$$

leads us to the theorem:

Theorem 11.3. ***Let*** $\omega(r)$ ***fulfill the condition***

$$\int_0^r \frac{\omega(t)}{t} dt \le C\,\omega(r).$$

Then the Calderon-Zygmund singular integral operator T ***with a standard kernel is bounded in the space*** ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$.

12 Riesz potential operator in the spaces $^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$

Theorem 12.1. *Let*

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} \alpha(x)p(x) < n. \tag{12.1}$$

Then

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \leq \frac{C}{t^{\frac{n}{p'(x_0)}}} \int\limits_0^t s^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))} ds \tag{12.2}$$

for every $f \in L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))$*, where*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{12.3}$$

Sobolev-Spanne type result; no Sobolev-Adams for these spaces.

Theorem 12.2. *Let conditions* (12.1) *and* (12.3) *fulfill and*

$$r^{\alpha(x_0)} \int\limits_0^r \frac{\omega_1(t)}{t} dt \leq C\, \omega_2(r).$$

Then the operator $I^{\alpha(\cdot)}$ *is bounded from* $^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ *to* $^{\complement}\mathcal{M}^{q(\cdot),\omega_2}_{\{x_0\}}(\Omega)$*.*