The Stable Fixed Point Property for Continuous Mappings in Metric Trees

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This Lecture

This lecture will be directed toward applications of some of the standard ideas of nonlinear analysis to the class of spaces calls $\mathbb{R}$-tree spaces.

Historical notes

Metric trees were first introduced by J. Tits ([1]). By using J. Tits's definitions of metric tree, it is shown that a metric space X is a complete metric tree if and only if X is hyperconvex with unique metric segments ([4]). Kirk introduced the fixed point property for continuous mappings in complete metric trees ([1]). Recently Kirk and Panyanak proved the KKM mapping principle in $\mathbb{R}$-trees. Some stability results in fixed point theory for Banach spaces state with S. Reich and A. J. Zaslavski . For various other generic aspects stability of fixed point theory, see ([1], [2], [3], [4]).

Our Results

We introduce the concept of the stable fixed point property for mappings, and we prove the existence of the stable fixed point property for continuous mappings defined on a compact and convex subset of a complete metric tree. Also, inspired of the Kirk's proof of the fixed point property for continuous mappings in complete metric trees, we generalize some theorems in such spaces.

Some Definitions

Let (X, d) be a metric space. A geodesic segment or a metric segment from x to y is the image of an isometric embedding $\alpha : [a, b] \to X$ of a closed interval $[a, b]$ of $\mathbb{R}$ such that $\alpha(a) = x$ and $\alpha(b) = y$. when there exists a unique geodesic segments from x to y, it will be denote by $[x, y]$. A geodesic ray is a subset of X isometric to the half-line $[0, \infty)$. For each $t \in [0, 1]$, let $(1 - t)x \oplus ty$ denote the unique member of $[x, y]$ satisfying

$$d(x, (1 - t)x \oplus ty) = td(x, y)$$

and

$$d((1 - t)x \oplus ty, y) = (1 - t)d(x, y).$$

Elementary Definitions

Definition

A $\mathbb{R}$-tree is a nonempty metric space X satisfying:

1. Any two points $x, y \in X$ are the endpoints of a unique metric segment $[x, y]$.
2. If $x, y, z \in X$ then $[x, y] \bigcap [x, z] = [x, w]$ for some $w \in X$ (i.e., if we have two metric segments with a common endpoint, then their intersection is a metric segment.)
3. If $x, y, z \in X$ and $[x, y] \bigcap [y, z] = \{y\}$ then $[x, y] \bigcup [y, z] = [x, z]$ (i.e., if two segments intersect in a single point, then their union is a metric segment.)

Example

Example

Consider $\mathbb{R}^2$ space with radial metric:

$$d_r(x,y) = \begin{cases} d(x,y) & \exists \lambda \in \mathbb{R}; x = \lambda y \\ d(x,\circ) + d(\circ,y) & otherwise. \end{cases}$$

The space $(\mathbb{R}^2, d_r)$ is a $\mathbb{R}$-tree.

Example

Example

Consider the complete $\mathbb{R}$-tree X obtained by taking all lines L_i in ℓ_2 of the form

$$L_i = \{te_i : -\infty < t < \infty\}$$

where e_i is the standard ith unit basis vector, and give X the shortest path metric.

Elementary Definitions

Definition

Let X be a metric tree and $A \subseteq X$. We say X is convex, if for all $x, y \in A$ we have $[x, y] \subseteq X$.

Definition

let X be a metric tree and $A \subseteq X$. A is called geodesically bounded if A does not contain a geodesic ray.

Some Elementary Results

Lemma

Let X be a $\mathbb{R}-$ trees space and let $x, y \in X$ such that $x \neq y$ and $s, t \in [0,1]$. Then $(1-t)x \bigoplus ty = (1-s)x \bigoplus sy$ if and only if $s = t$.

Some Elementary Results

Lemma

Let X be a metric tree and let $x, y \in X$ such that $x \neq y$. then

1. $[x, y] = \{(1-t)x \oplus ty | t \in [0,1]\}$.
2. $d(x,z) + d(z,y) = d(x,y)$ *if and only if* $z \in [x,y]$.
3. *The mapping* $f : [0,1] \longrightarrow [x,y], f(t) = (1-t)x \oplus ty$ *is continuous and bijective.*

Lemma

Let X be a metric tree. then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$$

for all $x, y, z \in X$ *and* $t \in [0,1]$.

Some Elementary Results

Lemma

Let (X, d) *be a metric tree,* $p, q, x, y \in X$ *and* $t \in [0, 1]$*. Then*

$$d((1-t)p \oplus tq, (1-t)x \oplus ty) \leq \max\{d(p, x), d(q, y)\}.$$

Kirk's Theorem

Theorem

Suppose X is a geodesically bounded complete $\mathbb{R}$-tree. Then every continuous mapping $f : X \longrightarrow X$ has a fixed point.

Theorem

Let $(X < d)$ be $\mathbb{R}$-tree, suppose K is a closed convex subset of X which does not contained a geodesic ray, suppose $intK \neq \emptyset$, and suppose $f : K \to X$ is continuous. Suppose there exists $p_0 \in intK$ such that $x \notin seg[p_0, f(x))$ for every $x \in \partial K$. Then f has a fixed point in K.

Note

Since any result proved for $CAT(0)$ spaces automatically carries over to $\mathbb{R} - trees$, we give some basic properties of metric segments in $CAT(0)$ spaces.

Elementary Definitions

Definition

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Examples

Example

Hilbert spaces (in which the $CAT(0)$-inequality is an equality) -the only Banach spaces that are $CAT(0)$.
$\mathbb{R}$-trees -the only hyperconvex metric spaces that are $CAT(0)$.

Kirk's Results

A subset K of metric space is said to have the approximate fixed point property (for nonexpansive mappings) if given any nonexpansive $f : K \longrightarrow K$, $\inf\{d(x, f(x)) : x \in K\} = 0$.
If X is a $CAT(0)$ space. Then X has the geodesic extension property if and only if every non-constant geodesic $c : [a, b] \longrightarrow X$ can be extended to a line $c : \mathbb{R} \longrightarrow X$.

Theorem

A closed convex subset of a complete $CAT(0)$ space with the geodesic extension property has the approximate fixed point property for nonexpansive mappings if and only if it does not contain a geodesic ray.

Kirk's Results

Theorem

Let K be a bounded closed convex subset of a complete $CAT(0)$ space X, let $f : K \longrightarrow K$ be nonexpansive, fix $x \in K$, and for each $t \in [0,1)$ let x_t be the point of $[x, f(x_t)]$ satisfying

$$d(x, x_t) = td(x, f(x_t)).$$

Then $\lim_{t \longrightarrow 1^-} x_t$ converges to the unique fixed point of f which is nearest x.

Kirk's Theorem

The existence of fixed points for nonexpansive mappings in a complete $CAT(0)$ space was proved by Kirk ([4]) as follows:

Theorem

Suppose K is a nonempty, bounded, closed and convex subset of complete $CAT(0)$space and suppose $f : K \longrightarrow K$ is nonexpansive . Then f has a fixed point.

Kirk's Theorem

Theorem

Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Suppose $f : K \longrightarrow X$ is nonexpansive mapping for which

$$\inf\{d(x, f(x)) : x \in K\} = 0$$

Then f has a fixed point in K.

Dinevari and Vaezpour's Results

Theorem

Suppose X is a complete $\mathbb{R}$-tree, C is a closed convex geodesically bounded subset of X and $T : C \longrightarrow C(X)$ is lower semicontinuous. Then there exists a $y_0 \in C$ such that

$$d(y_0, T(y_0)) = \inf_{x \in C} d(x, T(y_0)).$$

Moreover, if $y_0 \notin T(y_0)$ then $y_0 \in Bd(C)$.

Dinevari and Vaezpour's Results

Theorem

Suppose X is a complete $\mathbb{R}$-tree, C is a closed convex geodesically bounded subset of X and $T : C \longrightarrow X$ is continuous and for every $c \in X$ with $c \neq T(c)$,

$$(c, T(c)] := [c, T(c)] \backslash \{c\}$$

contains at least one point of C, then T has a fixed point.

Our Definitions

Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $\mathbb{R}$-tree (X, d). Consider

$$\mathcal{A} := \{A : A \textit{ is a continuous selfmap on } K \textit{ with } Fix(A) \neq \emptyset\},$$

with

$$\rho(A, B) = \sup\{d(Ax, Bx) : x \in K\}$$

for every $A, B \in \mathcal{A}$. Then $(\mathcal{A}, \rho)$ is a complete metric space.

Our Definitions

Definition

Let $K \subseteq X$ be a nonempty, bounded, closed and convex subset of a complete $\mathbb{R}$-tree (X, d). A map $A : K \to K$ is said to have the stable fixed point property if there exist $x \in Fix(A)$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni (B \in \mathcal{A}, \rho(\mathcal{A}, \mathcal{B}) \leq \delta) \Longrightarrow$$

$$\exists z \in K \ni (Bz = z, d(z, x) \leq \varepsilon).$$

Our Results

Theorem

Let (X, d) *be a complete* $\mathbb{R}$*-tree space,* $A \in \mathcal{A}$ *and* $\varepsilon > 0$*. then there exists* $\delta > 0$ *such that for each* $B \in \mathcal{A}$ *satisfying* $\rho(A, B) \leq \delta$ *and each* $x \in K$ *satisfying* $Bx = x$*, there exists* $y \in Fix(A)$ *such that* $d(x, y) \leq \varepsilon$*.*

The Question

In view of this result, It is natural to ask,does for each $A \in \mathcal{A}$, A have the stable fixed point property? Consider the next simple example:

Example

Example

Put $X := \mathbb{R}$, $K := [0, 1]$ and $Ax = x$ for all $x \in K$ so $Fix(A) = K$. and let $A_n x = (1 - \frac{1}{n})x$ and $B_n x = \min\{x + \frac{1}{n}, 1\}$ for all n. Therefore $A_n, B_n \to A$ and $Fix(A_n) = \{0\}$ and $Fix(B_n) = [1 - \frac{1}{n}, 1]$.

This example show that in general the answer too our question is negative. But in this talk we shall show that for a typical $A \in \mathcal{A}$ the answer of the question is positive.

Our Results

Theorem

Let (X, d) be a geodesically bounded normal complete $\mathbb{R}$-tree space, $A \in \mathcal{A}$, $\varepsilon > 0$ and $x \in Fix(A)$. Then there exist $B \in \mathcal{A}$ and $\delta > 0$ such that $\rho(A, B) \leq \varepsilon$ and $Bz = x$ for each $z \in K$ satisfying $d(z, x) \leq \delta$.

Our Results

Theorem

Let (X, d) be a geodesically bounded normal complete $\mathbb{R}$-tree space, $A \in \mathcal{A}$, $\varepsilon > 0$ and $x \in X$ be a fixed point of A. Let $B \in \mathcal{A}$ and $\delta > 0$ be as guaranteed by previous theorem. Then for each $C \in \mathcal{A}$ which $\rho(C, B) \leq \delta$, there exists $y \in K$ such that $Cy = y$ and $d(y, x) \leq \rho(B, C)$.

Our Results

Theorem

Let (X, d) be a geodesically bounded normal complete $\mathbb{R}$-tree space. Then there exists a subset $\mathcal{F}$ of $\mathcal{A}$ which is a countable intersection of open subsets of $(\mathcal{A}, \rho)$ so that for each $A \in \mathcal{F}$, A have the stable fixed point property.

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THANKS A LOT FOR YOUR ATTENTION

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