Mixed Problems for the Telegraph Equation in the Case of a System Consisting of Two Segments with Different Densities and Elasticities

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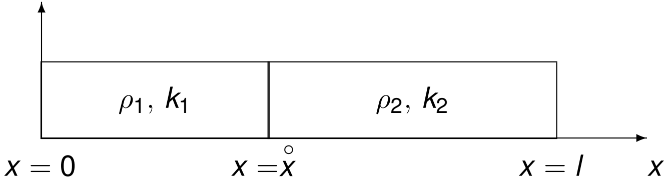
Formulation of the problem

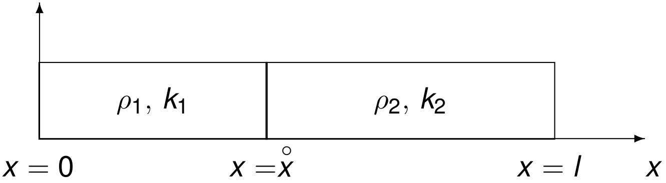
We consider the mixed problems for the longitudinal vibrations of a rod governed by the telegraph equation with Dirichlet and Neumann boundary conditions. The rod consists of two segments with different densities and elasticities.



- The rod has the linear density $\rho_1 = const$ and Young's modulus $k_1 = const$ on the segment $0 \leqslant x \leqslant \overset{\circ}{x}$.
- The rod has the linear density $\rho_2 = const$ and Young's modulus $k_2 = const$ on the segment $\overset{\circ}{x} \leqslant x \leqslant l$.

Formulation of the problem

- The impedances of these two segments are equal to each other.
- If the segments have different impedances, the point of their contact generates a series of reflected and transmitted (on both sides) waves, which are difficult to take into account completely. However, based on physics considerations, we can assume that this task simplifies considerably when the densities, elasticity coefficients, and lengths of the segments are such that a wave takes identical times to travel over each of them.



Mathamatical statement of the problem

$$u_{tt} = \begin{cases} a_1^2 u_{xx}(x,t) + c^2 u(x,t) & \text{in } Q_1 = [0 \leqslant x \leqslant \overset{\circ}{x}] \times [0 \leqslant t \leqslant T], \\ a_2^2 u_{xx}(x,t) + c^2 u(x,t) & \text{in } Q_2 = [\overset{\circ}{x} \leqslant x \leqslant l] \times [0 \leqslant t \leqslant T], \end{cases} \tag{1}$$

with zero initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{2}$$

with junction conditions at the joint point $\overset{\circ}{x}$,

$$u\left(\overset{\circ}{x} - 0, t\right) = u\left(\overset{\circ}{x} + 0, t\right), \tag{3^1}$$

$$k_1 u_x\left(\overset{\circ}{x} - 0, t\right) = k_2 u_x\left(\overset{\circ}{x} + 0, t\right) \tag{3^2}$$

and with one of the following three pairs of boundary conditions:

$$u(0,t) = \mu(t), \quad u(l,t) = \nu(t), \tag{4^1}$$

(displacements at two ends);

$$u_x(0,t) = \mu(t), \quad u_x(l,t) = \nu(t), \tag{4^2}$$

(elastic forces at two ends), and

$$u_x(0,t) = \mu(t), \quad u(l,t) = \nu(t), \tag{4^3}$$

(an elastic force at one end and a displacement at the other).

Definition

In the rectangle $Q = [0 \le x \le l] \times [0 \le t \le T]$, we consider the class (introduced in [Il'in, 2000]) $\hat{W}_2^1(Q)$,of functions $u(x,t)$ which are continuous in the closed rectangle Q and have generalized derivatives $u_x(x,t)$ and $u_t(x,t)$ that belong to $L_2(Q)$ and belong to $L_2[0 \le x \le l]$ for all $t \in [0,T]$ and to $L_2[0 \le t \le T]$ for all $x \in [0,l]$.

Denote the rectangles

$$Q_1 = [0 \le x \le \mathring{x}] \times [0 \le t \le T]$$

$$Q_2 = [\mathring{x} \le x \le l] \times [0 \le t \le T]$$

The integral identity

Solution is defined as a function $u(x,t)$ from the class $\hat{W}_2^1(Q)$ that satisfies the integral identity

$$\int_0^{\overset{\circ}{x}}\int_0^T u(x,t)\left[\Phi_{tt}(x,t)-a_1^2\Phi_{xx}(x,t)+c^2\Phi(x,t)\right]dxdt+$$

$$+\int_{\overset{\circ}{x}}^{l}\int_0^T u(x,t)\left[\Phi_{tt}(x,t)-a_2^2\Phi_{xx}(x,t)+c^2\Phi(x,t)\right]dxdt=$$

$$=-\int_0^{\overset{\circ}{x}} u(x,0)\Phi_t(x,0)dx-\int_{\overset{\circ}{x}}^{l} u(x,0)\Phi_t(x,0)dx+$$

$$+\int_0^{\overset{\circ}{x}} u_t(x,0)\Phi(x,0)dx+\int_{\overset{\circ}{x}}^{l} u_t(x,0)\Phi(x,0)dx+$$

$$
+\begin{cases} a_1^2\int\limits_0^T u(0,t)\Phi_x(0,t)dt - a_2^2\int\limits_0^T u(l,t)\Phi_x(l,t)dt & \text{for conditions } (4^1), \\ -a_1^2\int\limits_0^T u_x(0,t)\Phi(0,t)dt + a_2^2\int\limits_0^T u_x(l,t)\Phi(l,t)dt & \text{for conditions}(4^2), \\ -a_1^2\int\limits_0^T u_x(0,t)\Phi(0,t)dt - a_2^2\int\limits_0^T u(l,t)\Phi_x(l,t)dt & \text{for conditions } (4^3), \end{cases} \tag{5}
$$

The description of test functions $\Phi(x,t)$

for any test function $\Phi(x,t)$ that belongs to $C^{(2)}$ in each of the closed rectangles Q_1 and Q_2, with the transmission conditions

$$a_1\Phi(\mathring{x}-0,t) = a_2\Phi(\mathring{x}+0,t), \quad a_1^2\Phi_x(\mathring{x}-0,t) = a_2^2\Phi_x(\mathring{x}+0,t), \quad (**)$$

satisfies the zero final conditions $\Phi(x,T)\equiv 0$, $\Phi_t(x,T)\equiv 0$ and satisfies boundary conditions:

$$\begin{aligned} &\Phi(0,t)\equiv 0, \quad \Phi(l,t)\equiv 0 && \text{for conditions } (4^1), \\ &\Phi_x(0,t)\equiv 0, \quad \Phi_x(l,t)\equiv 0 && \text{for conditions } (4^2), \\ &\Phi_x(0,t)\equiv 0, \quad \Phi(l,t)\equiv 0 && \text{for conditions} (4^3). \end{aligned}$$

Equal impedances. Boundary conditions $u_x(0,t)=\mu(t), \quad u_x(l,t)=\nu(t)$

Let's define the functions

Definition

$$\underline{\mu}(t)=\begin{cases}0 \text{ for } t\leqslant 0,\\ \mu(t) \text{ for } t>0;\end{cases} \qquad \underline{\nu}(t)=\begin{cases}0 \text{ for } t\leqslant 0,\\ \nu(t) \text{ for } t>0;\end{cases}$$

Theorem

For $T>0$, $0<\mathring{x}<l$, $\mu(t),\nu(t)\in L_2[0,T]$ the mixed problem $(1),(2),(3^1)-(3^2)$ has a generalized solution $u(x,t)$ in $\widehat{W_2^1}(\mathrm{Q})$, which is given by the following formula:

Explicit formula. Equal impedances.

$$
u(x,t)=\begin{cases}
-a_1\sum\limits_{m=0}^{\infty}\int\limits_0^{t-\frac{x}{a_1}-2m\widehat{l}}\underline{\mu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\frac{x^2}{a_1^2}}\right)d\tau-\\
-a_1\sum\limits_{m=1}^{\infty}\int\limits_0^{t+\frac{x}{a_1}-2m\widehat{l}}\underline{\mu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\frac{x^2}{a_1^2}}\right)d\tau+\\
+a_2\sum\limits_{m=0}^{\infty}\int\limits_0^{t+\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}-2m\widehat{l}}\underline{\nu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}\right)^2}\right)d\tau+\\
+a_2\sum\limits_{m=1}^{\infty}\int\limits_0^{t-\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}-2m\widehat{l}}\underline{\nu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{x-\mathring{x}}{a_1}+\frac{l-\mathring{x}}{a_2}\right)^2}\right)d\tau & \text{in } Q_1,\\
-a_1\sum\limits_{m=0}^{\infty}\int\limits_0^{t-\frac{\mathring{x}}{a_1}-\frac{x-\mathring{x}}{a_2}-2m\widehat{l}}\underline{\mu}(\tau)J_0\left(c\sqrt{\left(t-\tau-2m\widehat{l}\right)^2-\left(\frac{\mathring{x}}{a_1}+\frac{x-\mathring{x}}{a_2}\right)^2}\right)d\tau-\\
-a_1\sum\limits_{m=1}^{\infty}\int\limits_0^{t+\frac{\mathring{x}}{a_1}+\frac{x-\mathring{x}}{a_2}-2m\widehat{l}}\underline{\mu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{\mathring{x}}{a_1}+\frac{x-\mathring{x}}{a_2}\right)^2}\right)d\tau+\\
+a_2\sum\limits_{m=0}^{\infty}\int\limits_0^{t-\frac{l-x}{a_2}-2m\widehat{l}}\underline{\nu}(\tau)J_0\left(c\sqrt{\left(t-\tau-2m\widehat{l}\right)^2-\left(\frac{l-x}{a_2}\right)^2}\right)d\tau+\\
+a_2\sum\limits_{m=1}^{\infty}\int\limits_0^{t+\frac{l-x}{a_2}-2m\widehat{l}}\underline{\nu}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{l-x}{a_2}\right)^2}\right)d\tau & \text{in } Q_2.
\end{cases}
$$

Equal travel times. Boundary conditions $u_x(0,t) = \mu_1(t), \quad u_x(l,t) = \nu_1(t)$

Let's define the functions

Definition

$$\underline{\mu}_1(t) = \begin{cases} 0 \text{ for } t \leqslant 0, \\ \mu_1(t) \text{ for } t > 0; \end{cases} \qquad \underline{\nu}_1(t) = \begin{cases} 0 \text{ for } t \leqslant 0, \\ \nu_1(t) \text{ for } t > 0; \end{cases}$$

Theorem

For $T > 0$, $0 < \mathring{x} < l$, $\mu_1(t), \nu_1(t) \in L_2[0,T]$ the mixed problem $(1), (2), (3^1) - (3^2)$ has a generalized solution $u_1(x,t)$ in $\widehat{W}_2^1(\mathrm{Q})$, which is given by the following formula:

Explicit formula in rectangle Q_1. Equal travel times.

$$u(x,t) = -a_1 \sum_{m=0}^{\infty} \int_0^{t-\frac{x}{a_1}-2\widehat{ml}} \underline{\mu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - \frac{x^2}{a_1^2}}\right) d\tau -$$

$$-a_1 \sum_{m=1}^{\infty} \int_0^{t+\frac{x}{a_1}-2\widehat{ml}} \underline{\mu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - \frac{x^2}{a_1^2}}\right) d\tau +$$

$$+\frac{a_1\rho_1 - a_2\rho_2}{a_1\rho_1 + a_2\rho_2}\Bigg\{ -a_1 \sum_{m=0}^{\infty} \int_0^{t+\frac{x}{a_1}-2\frac{\mathring{x}}{a_1}-2\widehat{ml}} \underline{\mu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - (\frac{x}{a_1}-2\frac{\mathring{x}}{a_1})^2}\right) d\tau -$$

$$-a_1 \sum_{m=1}^{\infty} \int_0^{t-\frac{x}{a_1}+2\frac{\mathring{x}}{a_1}-2\widehat{ml}} \underline{\mu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - (-\frac{x}{a_1}+2\frac{\mathring{x}}{a_1})^2}\right) d\tau \Bigg\} +$$

$$+\frac{2a_2\rho_2}{a_1\rho_1 + a_2\rho_2}\Bigg\{ a_2 \sum_{m=0}^{\infty} \int_0^{t+\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}-2\widehat{ml}} \underline{\nu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - \left(\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}\right)^2}\right) +$$

$$+a_2 \sum_{m=1}^{\infty} c\left(\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}\right) \int_0^{t-\frac{x-\mathring{x}}{a_1}+\frac{l-\mathring{x}}{a_2}-2\widehat{ml}} \underline{\nu_1}(\tau) J_0\left(c\sqrt{(t-\tau-2\widehat{ml})^2 - \left(\frac{x-\mathring{x}}{a_1}-\frac{l-\mathring{x}}{a_2}\right)^2}\right) d\tau \Bigg\}$$

$$u(x,t)=\frac{2a_1\rho_1}{a_1\rho_1+a_2\rho_2}\Bigg\{-a_1\sum_{m=0}^{\infty}\int_0^{t-\frac{\overset{\circ}{x}}{a_1}-\frac{x-\overset{\circ}{x}}{a_2}-2m\widehat{l}}\underline{\mu_1}(\tau)J_0\left(c\sqrt{\left(t-\tau-2m\widehat{l}\right)^2-\left(\frac{\overset{\circ}{x}}{a_1}+\frac{x-\overset{\circ}{x}}{a_2}\right)^2}\right)d\tau-$$

$$-a_1\sum_{m=1}^{\infty}\int_0^{t+\frac{\overset{\circ}{x}}{a_1}+\frac{x-\overset{\circ}{x}}{a_2}-2m\widehat{l}}\underline{\mu_1}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{\overset{\circ}{x}}{a_1}+\frac{x-\overset{\circ}{x}}{a_2}\right)^2}\right)d\tau\Bigg\}+$$

$$+a_2\sum_{m=0}^{\infty}\int_0^{t-\frac{l-x}{a_2}-2m\widehat{l}}\underline{\nu_1}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{l-x}{a_2}\right)^2}\right)d\tau+$$

$$+a_2\sum_{m=1}^{\infty}\int_0^{t+\frac{l-x}{a_2}-2m\widehat{l}}\underline{\nu_1}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{l-x}{a_2}\right)^2}\right)d\tau+$$

$$+\frac{a_2\rho_2-a_1\rho_1}{a_1\rho_1+a_2\rho_2}\times\Bigg[a_2\sum_{m=0}^{\infty}\int_0^{t-\frac{l+x-2\overset{\circ}{x}}{a_2}-2m\widehat{l}}\underline{\nu_1}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{l+x-2\overset{\circ}{x}}{a_2}\right)^2}\right)d\tau+$$

$$+a_2\sum_{m=1}^{\infty}\int_0^{t+\frac{l+x-2\overset{\circ}{x}}{a_2}-2m\widehat{l}}\underline{\nu_1}(\tau)J_0\left(c\sqrt{(t-\tau-2m\widehat{l})^2-\left(\frac{l+x-2\overset{\circ}{x}}{a_2}\right)^2}\right)d\tau\Bigg]$$

in rectangle Q_2

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Thanks for your attention!

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