Growth envelopes in weighted function spaces

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1.1 Growth envelopes

Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its **decreasing rearrangement** f^* be defined as usual,

$$f^*(t) := \inf\{s > 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leq t\}, \quad t > 0,$$

where $|\Omega|$ stands for the Lebesgue measure of a set Ω.

Definition

Let X be some quasi-normed function space on $\mathbb{R}^n$. The **growth envelope function** $\mathcal{E}_G^X : (0, \infty) \to [0, \infty]$ of X is defined by

$$\mathcal{E}_G^X(t) := \sup_{f \in X, ||f|X|| \leq 1} f^*(t), \qquad t > 0.$$

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Proposition

Let X, X_1, X_2 be (quasi-)normed function spaces on $\mathbb{R}^n$. Then

(i) $\mathcal{E}_{\mathsf{G}}^{X}$ *is monotonically decreasing and right-continuous,* $(\mathcal{E}_{\mathsf{G}}^{X})^* = \mathcal{E}_{\mathsf{G}}^{X}$.

(ii)

$$X \hookrightarrow L_\infty \qquad \Longleftrightarrow \qquad \mathcal{E}_{\mathsf{G}}^{X}(\cdot) \quad \text{is bounded.}$$

(iii)

$$X_1 \hookrightarrow X_2 \quad \Longrightarrow \quad \exists\, c > 0 \quad \forall\, t > 0 : \quad \mathcal{E}_{\mathsf{G}}^{X_1}(t) \leq c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t).$$

- „fine index" $u_{\mathsf{G}}^{X} \quad \rightsquigarrow \quad \mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^{X}(\cdot), u_{\mathsf{G}}^{X}\right)$ **growth envelope**
- $\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = \left(t^{-\frac{1}{p}}, q\right)$, $0 < p < \infty$, $0 < q \leq \infty$, $\quad \mathfrak{E}_{\mathsf{G}}(L_p) = \left(t^{-\frac{1}{p}}, p\right)$.

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Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. The **Hardy-Littlewood maximal operator** M is given by

$$Mf(x) := \sup_{B \in \mathcal{B}, x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

where $\mathcal{B}$ is the collection of all open balls.

Definition

Let $1 < p < \infty$ and $w \in L_1^{\text{loc}}(\mathbb{R}^n)$ positive a.e.

(i) w belongs to the **Muckenhoupt class** $\mathcal{A}_p$, if there exists a constant $0 < A < \infty$ such that for all balls $B \in \mathcal{B}$:

$$\left(\frac{1}{|B|} \int_B w(x) dx \right)^{1/p} \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p'/p} dx \right)^{1/p'} \leq A,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

1.2 Muckenhoupt weights

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Definition (continued)

(ii) $w \in \mathcal{A}_1$, if there exists an $0 < A < \infty$ such that for almost all $x \in \mathbb{R}^n$:

$$Mw(x) \leq Aw(x).$$

(iii)

$$\mathcal{A}_\infty := \bigcup_{p>1} \mathcal{A}_p.$$

- Muckenhoupt 1972, 1973
- $w(\Omega) = \int_\Omega w(x)dx, \quad \Omega \subset \mathbb{R}^n$ bounded, measurable set.
- $w \in \mathcal{A}_p,\ p \geq 1 \quad \Rightarrow \quad \forall\, \epsilon > 0 :\ w \in \mathcal{A}_{p+\epsilon}$
- $w \in \mathcal{A}_p,\ p > 1 \quad \Rightarrow \quad \exists\, \epsilon > 0 :\ w \in \mathcal{A}_{p-\epsilon}$
- $r_w := \inf \{r \geq 1 : w \in \mathcal{A}_r\}, \quad w \in \mathcal{A}_\infty.$
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Example

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & \text{if} \quad |x| \le 1, \\ |x|^\beta, & \text{if} \quad |x| > 1, \end{cases} \qquad \alpha,\ \beta > -n.$$

$$w_{\alpha,\beta} \in \mathcal{A}_p \quad \Leftrightarrow \quad \begin{cases} -n < \alpha, \beta < n(p-1), & \text{if} \quad 1 < p < \infty, \\ -n < \alpha, \beta \le 0, & \text{if} \quad p = 1. \end{cases}$$

Example

Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, $\varkappa \in \mathbb{R}$.

$$w_{\varkappa,\Gamma}(x) = \begin{cases} dist(x,\Gamma)^\varkappa, & \text{if} \quad dist(x,\Gamma) \le 1, \\ 1, & \text{if} \quad dist(x,\Gamma) > 1, \end{cases}$$

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- $w \in \mathcal{A}_\infty$, $0 < p < \infty$:
  The weigthed Lebesgue space $L_p(w) = L_p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f|L_p(w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty$$

- $Q_{\nu,m}$, $m \in \mathbb{Z}^n$, $\nu \in \mathbb{N}_0$: cubes with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$.

- $0 < p < \infty$, $m \in \mathbb{Z}^n$, $\nu \in \mathbb{N}_0$:

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hence $||\chi^{(p)}_{\nu,m}|L_p(\mathbb{R}^n)|| = 1$.

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Definition

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_\infty$.

(i) The weighted Besov space $B^s_{p,q}(w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f | B^s_{p,q}(w) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(w) \right\|^q \right)^{1/q}$$

is finite, (with the usual modification in the limiting case $q = \infty$).

(ii) The weighted Besov sequence space $b_{p,q}(w)$ is the collection of all complex-valued sequences $\lambda = \{\lambda_{\nu,m}\}_{\nu,m}$ such that

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2.1 Growth envelope for $B^s_{p,q}(w)$, $w \in \mathcal{A}_1$

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $n \max(\frac{1}{p} - 1, 0) < s < \frac{n}{p}$ *and* $w \in \mathcal{A}_1$ *with*

$$\inf_{m \in \mathbb{Z}^n} w(Q_{0,m}) \geq c > 0.$$

Then

(i)

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w)) = \mathfrak{E}_{\mathsf{G}}(B^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right).$$

(ii)

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim t^{-\frac{1}{p}}, \qquad t \to \infty.$$

- $s = \frac{n}{p}$, $1 < q \leq \infty$:

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2.2 Growth envelope for $B^s_{p,q}(w_{\alpha,\beta})$

Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$ *and* $\alpha_+ := \max(\alpha, 0)$. *Assume* $\alpha > -n$, $\beta \geq 0$ *and*

$$-n + \frac{\alpha_+}{p} < s - \frac{n}{p} < \frac{\alpha_+}{p}.$$

(i) *Then*

$$\mathfrak{E}_G(B^s_{p,q}(w_{\alpha,\beta})) = \left(t^{-\frac{1}{p}+\frac{s}{n}-\frac{\alpha_+}{np}}, q\right).$$

(ii) *Assume* $1 < p < \infty$ *and* $\alpha \leq \beta < n(p-1)$, *then*

$$\mathcal{E}_G^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\beta}{np}}, \qquad t \to \infty.$$

- Results also for limiting case
- Results also for F-case: Sobolev spaces

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- **Embeddings:** (estimate from above)

  Use:

$$X_1 \hookrightarrow X_2 \quad \Longrightarrow \quad \mathcal{E}_{\mathsf{G}}^{X_1}(t) \le c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t).$$

  Looking for:

$$B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}, \quad w \in \mathcal{A}_\infty.$$

  What conditions we have for the parameters?
  What restriction we have for the weight?

- **Atomic decomposition:** (estimate from below)

$$f_\lambda(x) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x),$$

  where $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in \mathbb{C}$ and $a_{\nu,m}$ are atoms.

2.4 Tools

- **Embeddings:** (estimate from above)

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- **Atomic decomposition:** (estimate from below)

$$f_\lambda(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x),$$

  where $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in \mathbb{C}$ and $a_{\nu,m}$ are atoms.

2.4 Tools

- **Embeddings:** (estimate from above)

  Use:
  $$X_1 \hookrightarrow X_2 \quad \Longrightarrow \quad \mathcal{E}_{\mathsf{G}}^{X_1}(t) \leq c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t).$$
  Looking for:
  $$B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}, \quad w \in \mathcal{A}_\infty.$$
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Definition

Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$ and $d > 1$.

The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s,p)_K$-atom, if for some $\nu \in \mathbb{N}_0$ and some $m \in \mathbb{Z}^n$ holds

$$\operatorname{supp} a \subset d\, Q_{\nu,m} \quad \text{and}$$
$$|D^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \quad \text{for} \quad |\alpha| \leq K, \quad x \in \mathbb{R}^n.$$

Let $\lfloor \alpha \rfloor := \max\{k \in \mathbb{Z} : k \leq \alpha\}$.

Proposition

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $K \in \mathbb{N}_0$ and $w \in \mathcal{A}_\infty$ with

$$K \geq (1 + \lfloor s \rfloor)_+ \quad \text{and} \quad n\left(\frac{r_w}{p} - 1\right)_+ < s;$$

then $f \in \mathcal{S}'$ belongs to $B^s_{p,q}(w)$ if, and only if, it can be written as

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x), \quad \text{converging in } \mathcal{S}',$$

where $a_{\nu,m}$ are $(s,p)_K$-atoms and $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in b_{p,q}(w)$; furthermore

$$||f|B^s_{p,q}(w)|| \sim \inf ||\lambda|b_{p,q}(w)||$$

where the infimum is taken over all admissible representations.

Motivation: $w \in \mathcal{A}_p \quad \Rightarrow \quad \exists\, c > 1, \quad \forall\, x \in \mathbb{R}^n, \forall\, r > 0:$

$$w(B(x,2r)) \leq cw(B(x,r)).$$

$\rightsquigarrow$ is called **doubling-property**.

$\rightsquigarrow$ A locally integrable, positive a.e. function w is called **doubling weight**, if w satisfied the **doubling property**.

$\rightsquigarrow$ Every Muckenhoupt weight is a **doubling weight**.

Question:

Is the class of the doubling weights strictly larger than $\mathcal{A}_\infty$?

Answer:

Yes, Example by Wik, 1989!

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Outlook

- Introduce $L_p(w)$, $B^s_{p,q}(w)$ with doubling weights w
- Atomic decomposition: Bownik [05, 07, 08]
- Wavelet characterisation: Bownik [03]
- Continuous and compact embeddings
- Entropy and approximation numbers
- Envelopes

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Thank you for your attention!