Nonlinear second order evolution equations with state-dependent delays

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The main topics of this talk

It is well known that the theory of monotone type operators can be applied to first order evolution equations and as particular cases to nonlinear functional parabolic equations of the form

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t, x, u, Du; u)] + a_0(t, x, u, Du; u) = f$$

where the last terms in the brackets mean "functional" (non-local) dependence on u, e.g. some integral operators applied to u or some state-dependent delays. It is less known that monotone type operators can be applied also to certain second order nonlinear evolution equations, including "functional" equations. The aim is to consider some second order evolution equations with functional dependence and state dependent delays. Differential equations and systems with state-dependent delay in one variable were considered thoroughly e.g. by I. Györi, F. Hartung, T. Krisztin, J. Turi, H.-O. Walther, J. Wu.

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0, T) \times \Omega$ and $p \geq 2$ be a real number. Let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of the usual Sobolev space $W^{1,p}(\Omega)$ (of real valued functions) containing $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$). Denote by $L^p(0, T; V)$ the Banach space of the set of measurable functions $u : (0, T) \to V$ with the norm

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|^p_V \, dt.$$

The dual space of $L^p(0, T; V)$ is $L^q(0, T; V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [9]).

We shall consider the equation

$$u'' + N(u'(t), u'([\gamma_0(u)](t))) + Qu + \tag{1}$$

$$M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))) = f$$

with the initial condition

$$u(0) = u_0, \quad u'(0) = u_1 \tag{2}$$

where $N : L^p(0,T;V) \times L^2(Q_T) \to L^q(0,T;V^\star)$ is a nonlinear operator, $(Qu)(t) = \tilde{Q}(u(t))$ and $\tilde{Q} : V \to V^\star$ is a linear and continuous operator,

$$M : L^p(0,T;V) \times L^2(Q_T) \times L^p_n(Q_T) \times L^2_n(Q_T) \to \times L^q(Q_T)$$

is a nonlinear operator.

Further, for $j = 0, 1, 2$
(G) $\gamma_j : L^2(Q_T) \to C_a[0, T]$ are continuous (nonlinear) operators such that

$$0 \le [\gamma_j(u)](t) \le t, \quad [\gamma_j(u)]'(t) \ge c_0$$

with some constant $c_0 > 0$.
Condition (G) is fulfilled e.g. by the operators of the form

$$[\gamma_j(u)(t) = t\beta\left(\int_{Q_t} \Gamma(t,\tau,\xi)u^2(\tau,\xi)d\tau d\xi\right)$$

where Γ, $\frac{\partial \Gamma}{\partial t}$ are continuous and nonnegative, $\beta \in C^1(\mathbb{R})$ satisfies $\delta_1 \le \beta \le 1$ with some constant $\delta_1 > 0$ and $\beta' \ge 0$.

(i) Assumptions on N:

$$N : L^p(0,T;V) \times L^2(Q_T) \to L^q(0,T;V^\star)$$

is bounded, demicontinuous and belongs to $(S)_+$ with respect to $D(L) = \{u \in L^p(0,T;V) : u' \in L^q(0,T;V^\star), u(0) = 0\}$, i.e.

$$(v_j) \to v \text{ weakly in } L^p(0,T;V), \quad v_j \in D(L),$$

$$(v_j') \to v' \text{ weakly in } L^q(0,T;V^\star), \quad (w_j) \to w \text{ (strongly) in } L^2(Q_T),$$

$$\limsup [N(v_j, w_j), v_j - v] \le 0$$

imply

$$(v_j) \to v \text{ (strongly) in } L^p(0,T;V).$$

Further, there are constants $c_2 > 0, c_3$ such that

$$[N(v,w), v] \ge c_2 \parallel v \parallel^p_{L^p(0,T;V)} - c_3.$$

(ii) Assumptions on Q: $(Qu)(t) = \tilde{Q}(u(t))$ and $\tilde{Q} : V \to V^\star$ is a linear and continuous operator,

$$\langle \tilde{Q}\tilde{u}, \tilde{v} \rangle = \langle \tilde{Q}\tilde{v}, \tilde{u} \rangle, \quad \langle \tilde{Q}\tilde{u}, \tilde{u} \rangle \geq 0, \quad \tilde{u}, \tilde{v} \in V.$$

(iii) Assumptions on M:

$$M : L^p(0, T; V) \times L^2(Q_T) \times L^p_n(Q_T) \times L^2_n(Q_T) \to \times L^q(Q_T)$$

is (nonlinear) bounded, demicontinuous and

$$\lim_{\|(u,\tilde{u},w,\tilde{w}\|)\to\infty} \frac{\| M(u, \tilde{u}, w, \tilde{w}) \|^q_{L^q(0,T;V^\star)}}{\| (u, \tilde{u}, w, \tilde{w}) \|^p} = 0.$$

Theorem

Assume (i) - (iii) and (G). Then for any $f \in L^q(0,T;V^\star)$, $u_0 \in V$ and $u_1 \in L^2(\Omega)$ there exists $u \in L^p(0,T;V)$ such that $u' \in L^p(0,T;V)$, $u'' \in L^q(0,T;V^\star)$ and u satisfies (1), (2).

Main steps of the proof For simplicity, consider the case $u_0 = 0, u_1 = 0$. Define operator $S : L^p(0,T;V) \to L^p(0,T;V)$ by

$$(Sv)(t) = \int_0^t v(s)ds.$$

Then S is a linear and continuous operator and u is a solution of (1), (2) with $u_0 = 0, u_1 = 0$ iff $v = u' \in L^p(0,T;V)$ satisfies

$$v' + N(v, v([\gamma_0(Sv)](t))) + QSv +$$

$$M(Sv, (Sv)([\gamma_1(Sv)]), DSv, (DSv)([\gamma_2(Sv)])) = f, \quad v(0) = 0.$$

Consider the operator $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ defined by

$$A(v) = N(v, v([\gamma_0(Sv)](t))) + QSv +$$

$$M(Sv, (Sv)([\gamma_1(Sv)](t)), DSv, (DSv)([\gamma_2(Sv)](t))).$$

It is not difficult to show that A is bounded and demicontinuous. Now we show that A belongs to $(S)_+$ with respect to

$$D(L) = \{v \in L^p(0, T; V) : v' \in L^q(0, T; V^\star), v(0) = 0\}$$

The last property means:

$$v_j \in D(L), \quad (v_j) \to v \text{ weakly in } L^p(0,T;V), \tag{3}$$

$$(v_j') \to v' \text{ weakly in } L^q(0,T;V^\star), \tag{4}$$

$$\limsup [A(v_j), v_j - v] \le 0 \tag{5}$$

imply

$$(v_j) \to v \text{ strongly in } L^p(0,T;V), \tag{6}$$

Lemma

Assume that $\gamma : L^2(Q_T) \to C_a[0,T]$ *satisfies (G). If* $(u_k) \to u$ *in* $L^2(Q_T)$ *and* $(w_k) \to w$ *in* $L^2(Q_T)$ *then*

$$w_k([\gamma(u_k)](t), x) \to w([\gamma(u)](t), x) \text{ in } L^2(Q_T).$$

Proof of the lemma Clearly,

$$w_k([\gamma(u_k)](t), x) - w([\gamma(u)](t), x) = \tag{7}$$

$$\{w_k([\gamma(u_k)](t), x) - w([\gamma(u_k)](t), x)\} +$$

$$\{w([\gamma(u_k)](t), x) - w([\gamma(u)](t), x)\}.$$

For the first term in the right hand side of (7) we have (by using the notation $\psi^k(t) = [\gamma(u_k)](t)$, (G))

$$\int_\Omega \left\{ \int_0^T w_k([\gamma(u_k)](t), x) - w([\gamma(u_k)](t), x)|^2 dt \right\} dx \le \tag{8}$$

$$\frac{1}{c_0} \int_\Omega \left\{ \int_0^T w_k(\psi_k(t), x) - w(\psi_k(t), x)|^2 \frac{\partial \psi_k}{\partial t} dt \right\} dx \le$$

$$\frac{1}{c_0} \int_{Q_T} |w_k(\tau, x) - w(\tau, x)|^2 d\tau dx \to 0.$$

Further, approximating the function $w \in L^2(Q_T)$ by a function $\tilde{w} \in C(\overline{Q_T})$, we we have for the second term on the right hand side of (7)

$$w([\gamma(u_k)](t), x) - w([\gamma(u)](t), x) = \tag{9}$$

$$\{w([\gamma(u_k)](t), x) - \tilde{w}([\gamma(u_k)](t), x)\} +$$

$$\{\tilde{w}([\gamma(u_k)](t), x) - \tilde{w}([\gamma(u)](t), x)\} +$$

$$\{\tilde{w}([\gamma(u)](t), x) - w([\gamma(u)](t), x)\}.$$

The first and third terms on the right hand side of (9) can be estimated similarly to (8). The $L^2(Q_T)$ norm of the second term on the right hand side of (9) is small for sufficiently large k because $\tilde{w}$ is uniformly continuous on $\overline{Q_T}$ and $(\gamma(u_k)) \to \gamma(u)$ in $C[0, T]$. So we have proved the lemma.

To prove that (3) - (5) imply (6), observe

$$[QS(v_j), v_j - v] = [QS(v_j - v), v_j - v] + [QS(v), v_j - v],$$

the first term on the right is nonnegative and the second term tends to 0, thus

$$\liminf [QS(v_j), v_j - v] \geq 0. \tag{10}$$

Further, by compact imbedding theorem, (3), (4) imply that $(v_j) \to v$ in $L^p(Q_T)$, for a subsequence, hence

$$[M(Sv_j, (Sv_j)([\gamma_1(Sv_j)]), DSv_j, D(Sv_j)([\gamma_2(Sv_j)])), v_j - v] \to 0 \tag{11}$$

because the first term in $[\cdot, \cdot]$ is bounded in $L^q(Q_T)$ since M is bounded.

(5), (10), (11) imply that

$$\limsup[N(v_j), v_j([\gamma_0(Sv_j)](t)), v_j - v] \leq 0$$

By the lemma

$$v_j([\gamma_0(Sv_j)](t)) \to v([\gamma_0(Sv)](t)) \text{ in } L^2(Q_T).$$

Thus (i) implies

$$(v_j) \to v \text{ in } L^p(0, T; V).$$

So $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is bounded, demicontinuous, belongs to $(S)_+$.
Finally, assumptions (i), (ii), (iii) imply:

$$\frac{[A(v), v]}{\| v \|^p} \geq \frac{c_2}{2} - \frac{c_4}{\| v \|^p},$$

thus A is coercive ($\lim_{\|v\| \to \infty} \frac{[A(v),v]}{\|v\|} = +\infty$). Consequently, there is a solution of (1), (2).

$$[N(v,w),z] = \sum_{i=1}^{n} \int_{Q_T} b(t,x,[H(w)](t,x))(D_i v)|Dv|^{p-2} D_i z dtdx +$$

$$\int_{Q_T} b_0(t,x,[H_0(w)](t,x)) v|v|^{p-2} z dtdx$$

where b, b_0 are Carathéodory functions, $0 < c_2 \leq b, b_0 \leq c_3$;

$H, H_0 : L^2(Q_T) \to C(\overline{Q_T})$ are continuous linear operators

$$\langle \tilde{Q}\tilde{u}, \tilde{v} \rangle = \int_\Omega \left[\sum_{k,l=1}^{n} a_{kl} D_k \tilde{u} D_l \tilde{v} + d_0 \tilde{u} \tilde{v} \right] dx$$

where $a_{kl}, d_0 \in L^\infty(\Omega)$, $a_{kl} = a_{lk}$, $\sum_{k,l=1}^{n} a_{kl}(x)\xi_k \xi_l \geq 0$, $d_0 \geq 0$.

$$M(u, \tilde{u}, w, \tilde{w}) =$$

$$\hat{b}(t, x, [F_1(\tilde{u})](t, x), [F_2(\tilde{w})](t, x)) \cdot \alpha(t, x, u, w)$$

where $\alpha, \hat{b}$ are Carathéodory functions

$$|\alpha(t, x, u, w)| \leq \text{const}[1 + |u|^\rho + |w|^\rho],$$

$$|\hat{b}(t, x, \theta_1, \theta_2)|^{q_1} \leq \text{const}[1 + \theta_1^2 + \theta_2^2]$$

where $0 \leq \rho < p - 1$, $q_1 = p/(p - 1 - \rho)$ and $F_j : L^2(Q_T) \to L^2(Q_T)$ are continuous operators satisfying with some $\sigma < p$

$$\int_{Q_T} |F_1(\tilde{u})|^2 \leq \text{const} \left[\int_{Q_T} |\tilde{u}|^2 \right]^{\sigma/2}, \int_{Q_T} |F_2(\tilde{w})|^2 \leq \text{const} \left[\int_{Q_T} |\tilde{w}|^2 \right]^{\sigma/2}$$

(For $p > 2$, σ may be 2, F_j linear continuous operator.)

It is possible and prove an existence theorem on solutions u for $t \in (0, \infty)$.
Further, one can formulate assumptions which imply the boundedness of $\| u'(t) \|_{L^2(\Omega)}$ and of $\| u(t) \|_{W^{1,2}(\Omega)}$ for $t \in (0, \infty)$.
Finally, some conditions imply

$$\lim_{t \to \infty} \| u'(t) \|_{L^2(\Omega)} = 0 \text{ and there exists } \lim_{t \to \infty} u(t) = w \in V.$$

First we formulate the existence theorem in $(0, \infty)$ which can be obtained from the existence Theorem for $(0, T)$, by using a diagonal process and the Volterra property (see, e.g. [8]). Denote by $L^p_{loc}(0, \infty; V)$ the set of functions $u : (0, \infty) \to V$ such that for each fixed finite $T > 0$, $u|_{(0,T)} \in L^p(0, T; V)$ and let $Q_\infty = (0, \infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ be the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T. On operators γ_j assume (G_∞) Operators $\gamma_j : L^2_{loc}(Q_\infty) \to C_a[0, \infty)$ are of Volterra type, i.e. $[\gamma_j(u)](T)$ depends only on $u|_{Q_T}$, for any finite T and $\gamma_j : L^2(Q_T)) \to C_a[0, T]$ is continuous for every T. Further,

$$\frac{\partial}{\partial t}[\gamma_j(u)](t, x) \geq c_0, \quad 0 \leq [\gamma_j(u)](t, x) \leq t$$

with some constant $c_0 > 0$.

Theorem

Assume that $\tilde{Q}: V \to V^\star$ *satisfies (ii). Let*

$$N: L^p_{loc}(0,\infty;V) \times L^2_{loc}(Q_\infty) \to L^q_{loc}(0,\infty;V^\star),$$

$$M: L^p_{loc}(0,\infty;V) \times L^2_{loc}(Q_\infty) \times L^p_{n,loc}(Q_\infty) \times L^2_{n,loc}(Q_\infty) \to L^q_{loc}(0,\infty;V^\star)$$

be operators of Volterra type and assume that for each finite $T > 0$ *their restrictions to* $(0,T)$ *satisfy (i) and (iii).*
Then for arbitrary $f \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in V$, $u_1 \in H$ *there exists* u *such that* $u \in C([0,\infty);V)$, $u' \in L^p_{loc}(0,\infty;V)$, $u'' \in L^q_{loc}(0,\infty;V^\star)$ *and*

$$u''(t) + N(u'(t), u'([\gamma_0(u)](t))) + Qu + \tag{12}$$

$$M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))) = f \text{ for a.a. } t \in (0,\infty),$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{13}$$

Theorem

Let the assumptions of the previous Theorem be satisfied such that for all $v \in L^p_{loc}(0,\infty;V)$, $w \in L^2_{loc}(Q_\infty)$

$$\langle N(v,w), v\rangle \geq c_2 \parallel v(t) \parallel_V^p, \quad t \in (0,\infty) \tag{14}$$

with some constant $c_2 > 0$ *and for all* $u \in L^p_{loc}(0,\infty;V)$, $\tilde{u} \in L^2_{loc}(Q_\infty)$, $w \in L^p_{n,loc}(Q_\infty)$, $\tilde{w} \in L^2_{n,loc}(Q_\infty)$

$$\parallel M(u,\tilde{u},w,\tilde{w}) \parallel_{V^\star}^q \leq \Phi_1(t), \quad t \in (0,\infty) \tag{15}$$

with some $\Phi_1 \in L^1(0,\infty)$. *Finally, let* $f \in L^q(0,\infty;V^\star)$. *Then for a solution* u *of (12), (13),* $y(t) = \parallel u'(t) \parallel_H^2$ *is bounded for* $t \in (0,\infty)$, $u' \in L^p(0,\infty;V)$ *and*

$$\langle \tilde{Q}[u(t)], u(t)\rangle \text{ is bounded for } t \in (0,\infty).$$

Now we formulate a theorem on the stabilization of the solution as $t \to \infty$. Assume that the assumptions of the previous Theorem are satisfied such that for all $v \in L^p_{loc}(0, \infty; V)$, $w \in L^2_{loc}(Q_\infty)$

$$\langle [N(v, w)](t), v(t) \rangle \geq c_2 (1 + t)^\mu \parallel v(t) \parallel_V^p, \quad t \in (0, \infty) \tag{16}$$

with some constants $\mu > p - 1$ ($p \geq 2$), $c_2 > 0$. Further, there exists $f_\infty \in V^\star$, a continuous function $\Phi \in L^1(0, \infty)$ with $\lim_\infty \Phi = 0$ such that

$$\parallel f(t) - f_\infty \parallel_{V^\star}^q \leq \Phi(t), \quad t \in (0, \infty) \tag{17}$$

and there exists a solution $u_\infty \in V$ of

$$\tilde{Q} u_\infty = f_\infty . \tag{18}$$

Then for a solution u of (12), (13) we have

$$\lim_{t\to\infty} \| u'(t) \|_H = 0, \tag{19}$$

$$\int_0^\infty (1+t)^\beta \| u'(t) \|_H^2 \, dt < \infty, \quad \int_0^\infty (1+t)^\mu \| u'(t) \|_V^p \, dt < \infty \tag{20}$$

where $0 \le \beta < [2\mu - (p-2)]/p$ and there exists $w \in V$ such that

$$\| u(t) - w \|_V^q \le \frac{\text{const}}{\lambda - 1} \frac{1}{(1+t)^{\lambda-1}} \text{ where } \lambda = \mu/(p-1) > 1. \tag{21}$$

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