One-sided compactness for interpolation of bilinear operators

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Introduction

- Multilinear operators appear naturally in several branches of classical harmonic analysis and functional analysis, including the theory of ideals of operators in Banach spaces.

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- "On Calderón's conjecture", Ann. Math. 149 (1999) 475–496

have shown the need for new results for bilinear operators.

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- It must be observed that the interpolation of bilinear operators is a classical problem in interpolation theory, and some results appear in the fundamental article of Lions-Peetre.

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- G. Pisier, "A remark on $\pi_2(\ell_2, \ell_2)$", Math. Nachrichten 148 (1990) 243–245,
- G. Botelho, C. Michels and D. Pellegrino, "Complex interpolation and summability properties of multilinear operators", Rev. Mat. Complut. 23 (2010) 139–161.

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Introduction

- We are interested in this essay in the behavior of compactness for bilinear operators under interpolation by the real method.
- The study on the behavior of linear compact operators under interpolation has its origin in the classical work of M. A. Krasnoselskii, for L^p spaces.
- Afterwards, several authors worked on the general question of compactness of operators for interpolation of abstract Banach spaces.

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- The first main results appeared in the works of J. L. Lions and J. Peetre, A. Calderón, A. Persson, S. G. Krein and Yu. I. Petunin and K. Hayakawa.

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- "Real and complex interpolation and extrapolation of compact operators", Duke Math. J. 65 No. 2 (1992) 333–343.
- In the case of the complex method, a similar result is still unknown.

Introduction

- Several researchers, as F. Cobos, M. Carro, L. Nikolova, D. L. Fernandez, J. Peetre, M. Mastylo, L. Maligranda, L. E. Persson, T. Khun, T. Schonbek, D. E. Edmunds, S. Janson and others, worked in the last two decades on compactness and related topics, for more general interpolation methods.

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- For interpolation of bilinear operators by the real method, the first abstract result is the following:

Introduction

- For the real method, if $\mathbf{E} = (E_0, E_1), \mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ are Banach couples, a classical result by Lions-Peetre assures that if

$$T : (E_0 + E_1) \times (F_0 + F_1) \to G_0 + G_1,$$

is bounded, with restrictions $T|E_k \times F_k \to G_k$, $(k = 0, 1)$ also bounded, then

$$T : \mathbf{E}_{\theta,p;J} \times \mathbf{F}_{\theta,q;J} \to \mathbf{G}_{\theta,r;J} \ ,$$

is bounded, where $0 < \theta < 1$ and $1/r = 1/p + 1/q - 1$.

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- M. Mastylo, "On interpolation of bilinear operators", J. Funct. Anal. 214 No. 2 (2004) 260–283.
- F. Cobos and L. M. Fernandez-Cabrera, "On the relationship between interpolation of Banach algebras and interpolation of bilinear operators", Canad. Math. Bull. 53 No. 1, (2010) 51–57.

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- Under an approximation hypothesis, Calderón established an one-side type general result, but restricted to interpolation spaces under the complex method.

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- For the multilinear case, the study on the behavior of compact operators in the interpolation spaces goes back to the classical paper of A. P. Calderón.
- Under an approximation hypothesis, Calderón established an one-side type general result, but restricted to interpolation spaces under the complex method.
- On the other hand, the behavior of compact multilinear operators under real interpolation functors until recently had not been investigated.

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- D. L. Fernandez and E. B. Silva, "Interpolation of bilinear operators and compactness", Nonlinear Analysis 73 (2010) 526–537.

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- D. L. Fernandez and E. B. Silva, "Interpolation of bilinear operators and compactness", Nonlinear Analysis 73 (2010) 526–537.
- generalizations for bilinear operators of Lions-Peetre compactness theorems (the one with the same departure spaces and the one with the same arriving spaces), Hayakawa's (i.e. a two-side result without approximation hypothesis) and a compactness theorem of Persson type were obtained.

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- The result is presented for the more general ρ method of interpolation.

Interpolation Spaces

- By a *function parameter* ρ we shall mean a continuous and positive function on $\mathbb{R}^+$. We shall say that a function parameter ρ belongs to the class $\mathcal{B}$, if it satisfies

  $\rho(1) = 1$ and,

  $\overline{\rho}(s) = \sup_{t>0} \frac{\rho(st)}{\rho(t)} < +\infty$, $s > 0$.

  Also, we shall say that a function parameter $\rho \in \mathcal{B}$ belongs to the class $\mathcal{B}^{+-}$ if it satisfies

$$\int_0^\infty \min(1, \frac{1}{t})\overline{\rho}(t)\frac{dt}{t} < +\infty.$$

Interpolation Spaces

- The function parameter $\rho_\theta(t) = t^\theta$, $0 \leq \theta \leq 1$, belongs to $\mathcal{B}$.

Interpolation Spaces

- The function parameter $\rho_\theta(t) = t^\theta$, $0 \leq \theta \leq 1$, belongs to $\mathcal{B}$.
- It corresponds to the usual parameter θ. Further, $\rho_\theta \in \mathcal{B}^{+-}$ if $0 < \theta < 1$, but ρ_0, $\rho_1 \notin \mathcal{B}^{+-}$.

Interpolation Spaces

- If E and F are intermediate spaces with respect to $\{E_0, E_1\}$ and $\{F_0, F_1\}$, respectively, we say that E and F are interpolation spaces of type ρ where $\rho \in \mathcal{B}^{+-}$, if given any $T \in L(\{E_0, E_1\}, \{F_0, F_1\})$ we have

$$||T||_{L(E,F)} \leq C||T||_0 \, \overline{\rho}\left(\frac{||T||_1}{||T||_0}\right), \qquad (1)$$

  where $||T||_k = ||T||_{L(E_k,F_k)}$, $(k = 0, 1)$ and $C > 0$ is a constant.

Interpolation Spaces

- Let $\{E_0, E_1\}$ be a Banach couple. The J and K functionals are defined by

$$J(t,x) = \max\{||x||_{E_0},\ t\ ||x||_{E_1}\},\ x \in E_0 \cap E_1,$$

$$K(t,x) = \inf_{x=x_0+x_1} \{||x_0||_{E_0} +\ t\ ||x_1||_{E_1}\},$$

respectively, where in $K(t,x)$, $x_0 \in E_0$ and $x_1 \in E_1$. Then, we can define the following interpolation spaces.

Interpolation Spaces

- The space $(E_0, E_1)_{\rho,q,K}$, $\rho \in \mathcal{B}$ and $0 < q \leq +\infty$, consists of all $x \in E_0 + E_1$ which norm

$$||x||_{\rho,q;K} = ||(\rho(2^n)^{-1} K(2^n, x; \mathbf{E}))_{n\in\mathbb{Z}}||_{\ell^q(\mathbb{Z})}$$

is finite.

Interpolation Spaces

- The space $(E_0, E_1)_{\rho,q;J}$, consists of all $x \in E_0 + E_1$, which it has a representation $x = \sum_{n=-\infty}^{\infty} u_n$ where $(u_n) \in E_0 \cap E_1$ and converges in $E_0 + E_1$, which norm

$$||x||_{\rho,q;J} = \inf ||(\rho(2^n)^{-1} J(2^n, u_n; \mathbf{E}))_{n\in\mathbb{Z}}||_{\ell^q(\mathbb{Z})},$$

is finite and where the infimum is taken over all representations $x = \sum u_n$.

Interpolation Spaces

- Given an intermediate space E with respect to a Banach couple $\mathbf{E} = (E_0, E_1)$ and $\rho \in \mathcal{B}^{+-}$, we say that E is an intermediate space of class $J_\rho(E_0, E_1)$ if

$$(E_0, E_1)_{\rho,1;J} \hookrightarrow E,$$

and we say that E is an intermediate space of class $K_\rho(E_0, E_1)$ if

$$E \hookrightarrow (E_0, E_1)_{\rho,\infty;K}.$$

Interpolation Spaces

- Given Banach couples $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$, we shall denote by $Bil(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ the set of all bilinear mappings $T : (E_0 + E_1) \times F_0 + F_1 \to G_0 + G_1$ such that $T|_{E_K \times F_k}$ is *bounded* from $E_k \times F_k$ into G_k, $k = 0, 1$, where

  $||T||_{Bil(E_k \times F_k, G_k)} = \sup\{||T(x,y)||_{G_k} :$
  $||x||_{E_k} \leq 1, ||y||_{F_k} \leq 1\}$.

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- The following definitions characterizes the bilinear interpolation functors which concerns to us.

Interpolation Spaces

Let $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0,\ F_1)$, and $\mathbf{G} = (G_0, G_1)$ be Banach couples. We shall say that $\mathcal{F}$ is a *bilinear interpolation functor of type* ρ, if $\mathcal{F}$ is an interpolation functor and if for all bilinear operator T from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$ whose restriction $T|_{E_k \times F_k}$ $(k = 0, 1)$ are bounded from $E_k \times F_k$ into G_k $(k = 0, 1)$ maps also $\mathcal{F}(E_0, E_1) \times \mathcal{F}(F_0, F_1)$ into $\mathcal{F}(G_0, G_1)$ and

$$||T||_{Bil(\mathcal{F}(\mathbf{E}) \times \mathcal{F}(\mathbf{F}); \mathcal{F}(\mathbf{G}))} \leq C ||T||_0\, \overline{\rho} \left(\frac{||T||_1}{||T||_0} \right)$$

Interpolation Spaces

Given Banach couples $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, \ F_1)$, and $\mathbf{G} = (G_0, G_1)$ and intermediate spaces E, F and G respectively, we shall say that the pair $(E \times F, G)$ is a *bilinear interpolation pair of type* ρ, if for all bilinear operator T from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$ such that $T : E \times F \to G$ one has

$$||T||_{Bil(E \times F; G)} \leq C ||T||_0 \, \overline{\rho} \left(\frac{||T||_1}{||T||_0} \right)$$

The following theorem from [FS] give us bilinear functors and interpolations pairs of type ρ.

Interpolation Spaces

- **Theorem 1.** Let T be a bounded bilinear operator from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$ whose restrictions $T|_{E_k \times F_k}$ $(k = 0, 1)$ are bounded from $E_k \times F_k$ into G_k $(k = 0, 1)$. Then, for $\rho \in \mathcal{B}^{+-}$, one has

$$T : E_{\gamma,p} \times F_{\rho,q} \to G_{\rho,r} \,,$$

where $\gamma(t) = \overline{\rho}(t^{-1})^{-1} \in \mathcal{B}^{+-}, 1/r = 1/p + 1/q - 1$ and

$$||T||_{Bil(E_{\gamma,p} \times F_{\rho,q}, G_{\rho,r})} \leq C\, ||T||_0\, \overline{\rho} \left(\frac{||T||_1}{||T||_0} \right) \,,$$

where $C > 0$ is a constant.

Interpolation Spaces

- Given Banach spaces E, F and G, a bounded bilinear mapping T from $E \times F$ into G is *compact* if the image of the set

$$M = \{(x, y) \in E \times F : \max\{||x||_E,\ ||y||_F\} \leq 1\}$$

is a totally bounded subset of G.

Main Results

- In this section we shall establish a bilinear version of Cwikel's compactness theorem for the ρ method, in which we assume compactness just in one of the departure spaces and any inclusion conditions.

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- In this section we shall establish a bilinear version of Cwikel's compactness theorem for the ρ method, in which we assume compactness just in one of the departure spaces and any inclusion conditions.
- We begin with a preliminary result which depends on an approximation hypothesis.

Main Results

- **Approximation Hypothesis.** A Banach couple $\mathcal{X} = (X_0, X_1)$ verifies the approximation hypothesis (AP) if there exists a sequence $\{P_n\}$ in $L(\mathcal{X}, \mathcal{X})$, with $P_n(X_0 + X_1) \subset X_0 \cap X_1$, and two other sequences $\{P_n^+\}$ and $\{P_n^-\}$ in $L(\mathcal{X}, \mathcal{X})$, such that

  (AP1) They are uniformly bounded in $L(\mathcal{X})$;
  (AP2) $I = P_n + P_n^+ + P_n^-, \quad n \in \mathbb{N}$;
  (AP3) $P_n^+ = P_n^+|_{X_0} \in L(X_0,\ X_1)$ and $P_n^- = P_n^-|_{X_1} \in L(X_1,\ X_0)$, and

$$\lim_{n\to\infty} ||P_n^+||_{L(X_0,\ X_1)} = \lim_{n\to\infty} ||P_n^-||_{L(X_1,\ X_0)} = 0.$$

Main Results

- **Theorem 2.** Let us assume that $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ are Banach couples which satisfy the approximation hypothesis (AP). Let T be a bounded bilinear operator such that the restrictions $T|_{E_k \times F_k}$ $(k = 0, 1)$ are bounded from $E_k \times F_k$ into G_k, $(k = 0, 1)$ and $T|_{E_0 \times F_0}$ is compact. Given $\rho \in \mathcal{B}^{+-}$, if $(E \times F, G)$ is a bilinear interpolations pair in respect to $(\mathbf{E} \times \mathbf{F}, \mathbf{G})$, then T is also compact from $E \times F$ into G.

Main Results

- Now, our main goal will be dealt with. We shall state a bilinear version of Cwikel's compactness theorem. The idea is to reduce it to Theorem 2.

Theorem 3. Let $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ be Banach couples. Let $T \in Bil(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ be given, such that the restriction $T|_{E_0 \times F_0}$ is compact from $E_0 \times F_0$ into G_0. Then, given $\rho \in \mathcal{B}^{+-}$, T is compact from $\mathbf{E}_{\gamma,p} \times \mathbf{F}_{\rho,q}$ into $\mathbf{G}_{\rho,r}$, where $\gamma(t) = 1/\overline{\rho}(t^{-1})$ and $1/r = 1/p + 1/q - 1$.

Main Results

- As a corollary, a bilinear version of Cwikel's theorem for the classical θ method is obtained.

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- **Corollary.** Let $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ be Banach couples. Let $T \in Bil(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ be given, such that the restriction $T|_{E_0 \times F_0}$ is compact from $E_0 \times F_0$ into G_0. Then, given $0 < \theta < 1$, T is compact from $\mathbf{E}_{\theta,p} \times \mathbf{F}_{\theta,q}$ into $\mathbf{G}_{\theta,r}$, where $1/r = 1/p + 1/q - 1$.

Main Results

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