On the Interplay of Regularity and Decay in Case of Radial Functions

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joint work with L. Skrzypczak (Poznan) and J. Vybiral (Linz)

• W. S. and L. Skrzypczak, *Radial subspaces of Besov and Lizorkin-Triebel classes: extended Strauss lemma and compactness of embeddings*, JFAA **6** (2000), 639-662.
• W. S., L. Skrzypczak and J. Vybiral, *On the interplay of regularity and decay in case of radial functions I. Inhomogeneous spaces*, Comm. in Cont. Math. (accepted).
• W. S. and L. Skrzypczak, *On the Interplay of regularity and decay in case of radial functions II. Homogeneous spaces*, Jena, Poznan, 2010.
• W. S., L. Skrzypczak and J. Vybiral, *Radial subspaces of Besov- and Lizorkin-Triebel spaces - complex interpolation and characterization by differences*, (in preparation), Jena, Poznan, Linz, 2011.

1. Introduction

The Radial Lemma (Strauss 1977).
Let $d \geq 2$. Every radial function $f \in H^1(\mathbb{R}^d)$ is almost everywhere equal to a function $\widetilde{f}$, continuous for $x \neq 0$, such that

$$|\widetilde{f}(x)| \leq c\,|x|^{\frac{1-d}{2}}\,\|\,f\,|H^1(\mathbb{R}^d)\|\,, \tag{1}$$

where c depends only on d.

This lemma contains three different assertions:

(a) the existence of a special representative of f, which is continuous outside the origin;

(b) the decay of f near infinity;

(c) the controlled unboundedness of f near the origin.

Compactness of embeddings

$$H^1(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d)\,, \qquad 2 \le q \begin{cases} \le \frac{2d}{d-2} & d > 2\,, \\ < \infty & d = 2\,. \end{cases}$$

Strauss (1977), Coleman, Glazer, Martin (1978) and Berestycki, P.L. Lions (1979):

$$RH^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L_q(\mathbb{R}^d)\,, \qquad 2 < q < \begin{cases} \frac{2d}{d-2} & d > 2\,, \\ \infty & d = 2\,. \end{cases}$$

Necessity of these restrictions for the compactness of the embedding: Ebihara, Schonbeck (1986).

3. The regularity of radial functions outside the origin

Theorem 1

Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Furthermore, we assume

$$s > \max\left(0, \frac{1}{p} - 1\right).$$

If $f \in RB^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$, then $f \in L_1(\mathbb{R}^d)$.

$$\mathrm{tr}\ f(t) := f(t, 0, \ldots, 0), \qquad t \in \mathbb{R}.$$

Theorem 2

Under the restrictions of Thm. 1 we have $\mathrm{tr}\ f \in B^s_{p,q}(\mathbb{R})$.

3.3 The continuity outside of the origin

Let $s > 1/p$ und $f \in RB^s_{p,q}(\mathbb{R}^d)$. Let φ be a smooth radial function s.t. $0 \notin \operatorname{supp} \varphi$ and

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |D^\alpha \varphi(x)| < \infty \, .$$

Then the pointwise product $\varphi \cdot f \in RB^s_{p,q}(\mathbb{R}^d)$ and

$$\operatorname{tr}(\varphi \cdot f) \in B^s_{p,q}(\mathbb{R}) \hookrightarrow B^{s-1/p}_{\infty,\infty}(\mathbb{R}) = \mathcal{Z}^{s-1/p}(\mathbb{R}) \, .$$

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s-d/p}_{\infty,\infty}(\mathbb{R}^d) = \mathcal{Z}^{s-d/p}(\mathbb{R}^d)$$

Corollary 1

Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s > 1/p$. Let φ be as above. If $f \in RB^s_{p,q}(\mathbb{R}^d)$, then $\varphi f \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$ follows.

Corollary 2

Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. If either $s > 1/p$ (and q arbitrary) or $s = 1/p$ and $q \leq 1$, then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on $|x| > \tau$.

- P.L. Lions (1982) (Sobolev spaces), S. and Skrzypczak (2000);
- $U := \{(s,p,q) : \quad (s,p,q)$ as in Cor. 2$\}$.

4. Decay of radial functions near infinity. I

Theorem 3

Let $d \geq 2$, $0 < p < \infty$, *and* $0 < q \leq \infty$. *Furthermore we assume* $(s,p,q) \in U$. *Then there exists a constant* c *s.t.*

$$|x|^{(d-1)/p}\, |f(x)| \leq c \, \| f \, | B^s_{p,q}(\mathbb{R}^d) \|$$

holds for all $|x| \geq 1$ *and all* $f \in RB^s_{p,q}(\mathbb{R}^d)$. *Moreover*

$$\lim_{|x| \to \infty} |x|^{\frac{d-1}{p}}\, |f(x)| = 0$$

holds for all $f \in RB^s_{p,q}(\mathbb{R}^d)$.

Theorem 4

(i) *Let* $(s,p,q) \in U$*. Then there exists a constant* $c > 0$ *s.t. for all* x*,* $|x| > 1$ *there exists a smooth radial function* $f \in RB^s_{p,q}(\mathbb{R}^d)$ *with* $\| f \, |RB^s_{p,q}(\mathbb{R}^d)\| = 1$*, and satisfying*

$$|x|^{\frac{d-1}{p}} \, |f(x)| \ge c \, . \qquad (2)$$

(ii) *Let* $(s,p,q) \notin U$ *and* $\frac{1}{p} > \sigma_p(d)$*. For all sequences* $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ *with* $\lim_{j\to\infty} |x^j| = \infty$ *there exists a radial function* $f \in RB^s_{p,q}(\mathbb{R}^d)$*,* $\| f \, |RB^s_{p,q}(\mathbb{R}^d)\| = 1$*, s.t.* f *is unbounded in any neighborhood of* x^j*,* $j \in \mathbb{N}$*.*

Remark

Let p be fixed. Then increasing the smoothness s will not result in an improved decay rate.

5. Controlled unboundedness of radial functions near the origin

Theorem 5

Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$.
(i) *Let $(s, p, q) \in U$ and $s < \frac{d}{p}$. Then there exists a constant $c > 0$ s.t.*

$$|x|^{\frac{d}{p}-s}\,|f(x)| \leq c\,\|\, f\,|RB^s_{p,q}(\mathbb{R}^d)\| \tag{3}$$

holds for all $0 < |x| \leq 1$ and all $f \in RB^s_{p,q}(\mathbb{R}^d)$.
(ii) *Let $\sigma_p(d) < s < d/p$. Then there exists a constant $c > 0$ s.t. for all x, $0 < |x| < 1$, there is a smooth radial function $f \in RB^s_{p,q}(\mathbb{R}^d)$, $\|\, f\,|RB^s_{p,q}(\mathbb{R}^d)\| = 1$, satisfying*

$$|x|^{\frac{d}{p}-s}\,|f(x)| \geq c\,. \tag{4}$$

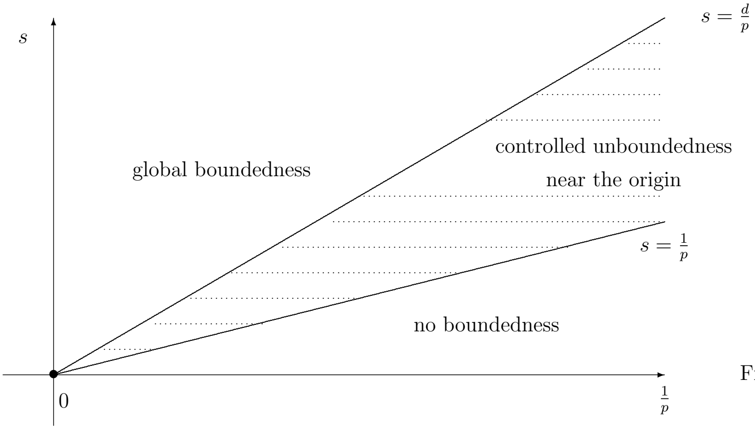
The limiting case

Theorem 6

Let $d \geq 2$, $0 < p < \infty$, $1 < q \leq \infty$ and $s = d/p$. Then there esists a constant $c > 0$ s.t.

$$(-\log|x|)^{-1/q'}\,|f(x)| \leq c\,\| f\,|B^{d/p}_{p,q}(\mathbb{R}^d)\|$$

holds for all $0 < |x| \leq 1/2$ and all $f \in RB^{d/p}_{p,q}(\mathbb{R}^d)$.


$s = \frac{d}{p}$
s
controlled unboundedness
near the origin
global boundedness
$s = \frac{1}{p}$
no boundedness
0
$\frac{1}{p}$
Fig. 1

6. Compact embeddings

Theorem 7

Let $d \geq 2$ and $0 < p_0, p_1, q_0, q_1 \leq \infty$. Then $RB^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ is compactly embedded into $B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ if, and only if,

$$p_0 < p_1 \qquad \text{and} \qquad s_0 - s_1 > d\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Remark

S./Skrzypczak (2000). Use of adapted atomic decompositions and associated sequence spaces.

Corollary 3

Let $d \geq 2$ and $1 \leq p_1 \leq \infty$. Then $RB^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ is compactly embedded into $L_{p_1}(\mathbb{R}^d)$ if, and only if,

$$0 < q_0 \leq \infty, \qquad p_0 < p_1 \qquad \textit{and} \qquad s_0 > d\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

- $RB^1_{2,2}(\mathbb{R}^d) = RH^1(\mathbb{R}^d)$ in the sense of equivalent norms.

- S. and Skrzypczak (2000) (many forerunners in case of Sobolev spaces).
- Triebel (2006; different proof).
- Cwikel and Tintarev (2011; different proof).

$s_0 < d/p_0$:

$$s_0 > d\left(\frac{1}{p_0} - \frac{1}{p_1}\right) \quad \Longleftrightarrow \quad p_1 < \frac{d}{\frac{d}{p_0} - s_0}\,.$$

Definition

Let $c > 0$. The set $B^s_{p,q}(\mathbb{R}^d, \mathrm{sub}\, c)$ is the collection of all functions $f \in B^s_{p,q}(\mathbb{R}^d)$ s.t. there exists a $g \in RB^s_{p,q}(\mathbb{R}^d)$ satisfying the inequalities

$$\begin{aligned} |f(x)| &\le |g(x)| \qquad a.e.\ in \quad \mathbb{R}^d\,; \\ \|\, g\,|B^s_{p,q}(\mathbb{R}^d)\| &\le c\,\|\, f\,|B^s_{p,q}(\mathbb{R}^d)\|\,. \end{aligned}$$

Theorem 8

Let $d \geq 2$. *Let* $0 < p_0 \leq \infty$, $1 \leq p_1 \leq \infty$, $0 < q_0 \leq \infty$ *and*

$$\max\left(\frac{1}{p_0}\,,\,\frac{d}{p_0} - d\right) < s_0 < \frac{d}{p_0}\,.$$

Let $c > 0$. *Then* $B^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mathrm{sub}\, c)$ *is compactly embedded into* $L_{p_1}(\mathbb{R}^d)$ *if, and only if,*

$$p_0 < p_1 < \frac{d}{\frac{d}{p_0} - s_0}\,.$$

Lemma 1

Let $d \geq 2$. Let $0 < p_0 \leq \infty$, $1 \leq p_1 \leq \infty$, $0 < q_0 \leq \infty$ and $s_0 = d/p_0$. Let $c > 0$. Then $B^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mathrm{sub}\, c)$ is compactly embedded into $L_{p_1}(\mathbb{R}^d)$ if, and only if, $p_0 < p_1 < \infty$.

Lemma 2

Let $1 \leq p_1 \leq \infty$. For all admissible parameters d, s_0, p_0, q_0 the set

$$\bigcup_{c>0} B^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mathrm{sub}\, c)$$

is not compactly embedded into $L_{p_1}(\mathbb{R}^d)$.

Theorem 9

Let $d \geq 2$*. Let* $0 < p_0, q_0 \leq \infty$, $0 < p < \infty$, $1 \leq p_1 \leq \infty$ *and*

$$s_0 > d \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right).$$

Let $c > 0$*. Then* $B^{s_0}_{p_0,q_0}(\mathbb{R}^d) \cap B^{1/p}_{p,1}(\mathbb{R}^d, \mathrm{sub}\, c)$ *is compactly embedded into* $L_{p_1}(\mathbb{R}^d)$*.*

Definition

We suppose

$$\sigma_p < s < \frac{d}{p}\,.$$

Let $c > 0$. *The set* $\dot{B}^s_{p,q}(\mathbb{R}^d, \mathrm{sub}\, c)$ *is the collection of all subradial functions* $f \in B^s_{p,q}(\mathbb{R}^d)$ *s.t. there exists a*

$$g \in \dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)\,, \qquad t := \frac{d}{\frac{d}{p} - s}\,,$$

satisfying the inequalities

$$\begin{aligned} |f(x)| &\le |g(x)| \qquad \text{a.e. in} \quad \mathbb{R}^d\,; \\ \| g\,|\dot{B}^s_{p,q}(\mathbb{R}^d)\| &\le c\, \|\, f\,|B^s_{p,q}(\mathbb{R}^d)\|\,. \end{aligned}$$

Theorem 10

Let $d \geq 2$. *Let* $0 < p_0, q_0 \leq \infty$, $1 - \frac{1}{d} < p < \infty$, $1 \leq p_1 \leq \infty$ *and*

$$s_0 > d \, \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right).$$

Let $c > 0$. *Then* $B^{s_0}_{p_0,q_0}(\mathbb{R}^d) \cap \dot{B}^{1/p}_{p,1}(\mathbb{R}^d, \mathrm{sub}\, c)$ *is compactly embedded into* $L_{p_1}(\mathbb{R}^d)$.

Remark

$$1 - \frac{1}{d} < p < \infty \quad \Longleftrightarrow \quad d \, \max\left(\frac{1}{p} - 1\right) < \frac{1}{p}.$$