Solution to the Navier–Stokes equations with random initial data

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Representations of the Navier–Stokes equations

Consider the classical Navier–Stokes equations:

$$\begin{cases} \frac{\partial}{\partial t} y(t,\theta) = -(y,\nabla)y(t,\theta) + \nu\Delta y(t,\theta) - \nabla p(t,\theta), \\ \operatorname{div} y = 0, \\ y(0,\theta) = y_0(\theta). \end{cases}$$

$\theta \in \mathbb{T}^d$, $d \geqslant 2$, (d-dimensional torus), $t \in [0, T]$

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- Galerkin-type equations and their solution. Absence of a blow-up time
- Convergence of the series representing the non-linear term in the Navier–Stokes equations in $L_2(H^\alpha, \gamma)$, where γ is a specially introduced Gaussian measure on H^α
- Convergence in distribution of the Galerkin-type approximations to the Navier–Stokes solution (by means of application of the tightness criteria for probability measures and the Skorokhod theorem)

Representation of the Navier–Stokes equations

Notation

$$\mathbb{Z}_d^+ = \{(k_1, k_2, \dots, k_d) \in \mathbb{Z}_d : k_1 > 0 \text{ or } k_1 = \dots = k_{i-1} = 0,\ k_i > 0,$$
$$i = 2, \dots, d\};$$

if $k = (k_1, \dots, k_d) \in \mathbb{Z}_d^+$, and $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$,

then $|k| = \sqrt{\sum_{i=1}^d k_i^2}$, $k \cdot \theta = \sum_{i=1}^n k_i \theta_i$, $\Delta_{\mathbb{T}^d} = \sum_{i=1}^d \frac{\partial^2}{\partial \theta_i^2}$.

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For every $k \in \mathbb{Z}_d^+$, $(\bar{k}^1, \dots, \bar{k}^{d-1})$ denotes an orthogonal system of vectors of length $|k|$ which is also orthogonal to k.

Representation of the Navier–Stokes equations

The Fourier series expansion of a divergence-free vector field on $\mathbb{T}^d$:

$$\sum_{p=1}^{d} u_0^p\, e^p + \sum_{k\in\mathbb{Z}_d^+} \sum_{p=1}^{d-1} \Big[u_k^p\, A_k^p + v_k^p\, C_k^p \Big]$$

where

$$A_k^p = \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \cos(k\cdot\theta)\, \frac{\bar{k}^p}{|k|}, \quad C_k^p = \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \sin(k\cdot\theta)\, \frac{\bar{k}^p}{|k|},$$

$$p = 1, \ldots, d-1,\ k \in \mathbb{Z}_d^+,$$

and the constant vector fields e^p, $p = 1, \ldots, d$, are such that the p-th coordinate is $\frac{1}{(2\pi)^{\frac{d}{2}}}$ and the other coordinates are 0.

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$$\sum_{k\in\mathbb{Z}_d^+} \sum_{p=1}^{d-1} |k|^{2\alpha}((u_k^p)^2 + (v_k^p)^2) < \infty.$$

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- We will search the Navier–Stokes solution in the form:

$$y(t,\theta) = \sum_{p=1}^{d} u_0^p(t)\, e^p + \sum_{k\in\mathbb{Z}_d^+} \sum_{p=1}^{d-1} \Big[u_k^p(t)\, A_k^p + v_k^p(t)\, C_k^p \Big].$$

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- Define $u(t) = \{u_k^p(t), v_k^p(t), u_0^q(t)\}$, $k \in \mathbb{Z}_d^+$, $p = 1, \dots, d-1$, $q = 1, \dots, d$

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- $\mathbf{P}$ is the projector onto the divergence-free vector fields
- By $\Delta u(s)$ we understand the vector with the coordinates

$$\{-|k|^2 u_k^p(s), -|k|^2 v_k^p(s),\ k \in \mathbb{Z}_d^+,\ p = 1,\dots,d-1\}$$

Representation of the Navier–Stokes equations

- The above equation in the coordinate form:

$$
\begin{aligned}
u_0^p(t) &= u_0^p(0), \\
u_k^p(t) &= u_k^p(0) - \int_0^t B_k^{p,\cos}(u(s))\, ds - \nu |k|^2 \int_0^t u_k^p(s)\, ds, \\
v_k^p(t) &= v_k^p(0) - \int_0^t B_k^{p,\sin}(u(s))\, ds - \nu |k|^2 \int_0^t v_k^p(s)\, ds,
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- Define

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u_k(t) = \sum_{p=1}^{d-1} u_k^p(t) \frac{\bar{k}^p}{|k|}, \quad v_k(t) = \sum_{p=1}^{d-1} u_k^p(t) \frac{\bar{k}^p}{|k|}, \quad u_0(t) = \sum_{p=1}^{d} u_0^p(t)\, e^p.
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- $y(t,\theta) = u_0(0) + \sum_{k\in\mathbb{Z}_d^+} \Big[u_k(t)\cos(k\cdot\theta) + v_k(t)\sin(k\cdot\theta)\Big].$

Representation of the Navier–Stokes equations

- Define $B_k^{\cos} = \sum_{p=1}^{d-1} B_k^{p,\cos} \frac{\bar{k}^p}{|k|}, \quad B_k^{\sin} = \sum_{p=1}^{d-1} B_k^{p,\sin} \frac{\bar{k}^p}{|k|}.$

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- The Navier–Stokes equations take the form:

$$u_k(t) = u_k(0) - \int_0^t B_k^{\cos}(u(s))\,ds - \nu|k|^2 \int_0^t u_k(s)\,ds$$
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- Equivalent form:

$$u_k(t) = e^{-t\nu|k|^2} u_k(0) - \int_0^t e^{-(t-s)|k|^2\nu} B_k^{\cos}(u(s))\, ds$$
$$v_k(t) = e^{-t\nu|k|^2} v_k(0) - \int_0^t e^{-(t-s)|k|^2\nu} B_k^{\sin}(u(s))\, ds.$$

Galerkin-type approximations

- The components of the non-linear term are represented by infinite series:

$$B_k^{\sin}(u) = \sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k,h) \left(u_{k+h}^i u_h^j + v_{k+h}^i v_h^j \right) \mathbf{P}_k \bar{h}^j$$

$$+ \lambda_i^-(k,h) \left(\operatorname{sign}(k-h) v_{\pm(k-h)}^i v_h^j - u_{\pm(k-h)}^i u_h^j \right) \mathbf{P}_k \bar{h}^j - \sum_{i=1}^{d} \lambda_i(k) u_0^i \sum_{j=1}^{d-1} u_k^j \, \bar{k}^j .$$

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- Similar for $B_k^{\sin}(u)$
- In the space H^α of u_k^i, v_k^i, we introduce a Gaussian measure γ so that the infinite series for $B_k^{\sin}$ and $B_k^{\cos}$ converge in $L_2(H^\alpha, \gamma)$.

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- In the space H^α of u_k^i, v_k^i, we introduce a Gaussian measure γ so that the infinite series for $B_k^{\sin}$ and $B_k^{\cos}$ converge in $L_2(H^\alpha, \gamma)$.
- Namely,

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- For any $\alpha \in \mathbb{R}$, we define a Hilbert space

$$H^\alpha = \Big\{ u : \sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} \big(|u_k|^2 + |v_k|^2\big) < \infty \Big\}.$$

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- For each $k \in \mathbb{Z}_d^+$, we define the Gaussian measure γ_k on $\mathbb{R}^{2(d-1)}$

$$\left(\frac{|k|^{2l}}{2\pi} \right)^{(d-1)} \exp\left(-\frac{|k|^{2l}}{2} (|u_k|^2 + |v_k|^2) \right).$$

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$$\gamma(du) = \bigotimes_{k \in \mathbb{Z}_d^+} \gamma_k \big(d(u_k, v_k) \big).$$

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- **Lemma**

Let $l > \alpha + \frac{d}{2} + 1$. Then, $B_k^{\sin}, B_k^{\cos} \in L_2(\gamma, \mathbb{R}^{d-1})$, and $B \in L_2(\gamma, H^\alpha)$.

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- For $|k| \leqslant n$, let $B_k^{(n),\sin}(s,u)$ and $B_k^{(n),\cos}(s,u)$ be obtained from $B_k^{\sin}(s,u)$ and $B_k^{\cos}(s,u)$ by restricting the summation over $|h| \leqslant n$.

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- $\tilde{u}_k(t) = e^{t\nu|k|^2} u_k(t)$, $\tilde{v}_k(t) = e^{t\nu|k|^2} v_k(t)$, $\tilde{u}(s) = \{\tilde{u}_k(t), \tilde{v}_k(t)\}$, $k \in \mathbb{Z}_d^+$.

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- $\tilde{B}_k^{(n),\sin}(s,\tilde{u}(s)) = e^{s\nu|k|^2} B_k^{(n),\sin}(u(s))$,
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- $\tilde{B}^{(n)} = \sum_{k \in \mathbb{Z}_d^+} \tilde{B}_k^{(n),\cos} \cos(k \cdot \theta) + \tilde{B}_k^{(n),\sin} \sin(k \cdot \theta)$
  $B^{(n)} = \sum_{k \in \mathbb{Z}_d^+} B_k^{(n),\cos} \cos(k \cdot \theta) + B_k^{(n),\sin} \sin(k \cdot \theta)$.

Galerkin-type approximations

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where $u \in H^\alpha$.

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- **Lemma**

The above ODE has a unique solution $\tilde{U}^n(t,u)$ on $[0,T]$ for γ-almost all initial conditions $u \in H^\alpha$. Measure γ is quasi-invariant with respect to $\tilde{U}^n(t,u)$.

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  where $u \in H^\alpha$.

- **Lemma**

  The above ODE has a unique solution $\tilde{U}^n(t,u)$ on $[0,T]$ for γ-almost all initial conditions $u \in H^\alpha$. Measure γ is quasi-invariant with respect to $\tilde{U}^n(t,u)$.

- To prove that there is no blow-up time we applied the results of

  A. B. Cruzeiro, *Equations différentielles ordinaires: non explosion et mesures quasi-invariantes*, J. Funct. Anal. 54 (1983), 193–205.

Navier–Stokes solution as the limit of Galerkin-type approximations

- Denote by $\tilde{U}_k^n(t,u)$ and $\tilde{V}_k^n(t,u)$ the components of the solution $\tilde{U}^n(t,u)$

Navier–Stokes solution as the limit of Galerkin-type approximations

- Denote by $\tilde{U}_k^n(t,u)$ and $\tilde{V}_k^n(t,u)$ the components of the solution $\tilde{U}^n(t,u)$
- Define $U_k^n(t,u) = e^{-t\,|k|^2\,\nu}\,\tilde{U}_k^n(t,u), \quad V_k^n(t,u) = e^{-t\,|k|^2\,\nu}\,\tilde{V}_k^n(t,u)$

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- Consider the triple $(H^\alpha, \mathcal{B}, \gamma)$ as a probability space, and $U^n(t,u) = \{U_k^n(t,u), V_k^n(t,u)\}$, $k \in \mathbb{Z}_d^+$, as a stochastic process on it.

Navier–Stokes solution as the limit of Galerkin-type approximations

- Denote by $\tilde{U}^n_k(t,u)$ and $\tilde{V}^n_k(t,u)$ the components of the solution $\tilde{U}^n(t,u)$
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- **Lemma**

  There exists a subset $H' \subset H^\alpha$ of full γ-measure such that for all $u \in H'$, for all $t \in [0,T]$, $U^n(t,u)$ verifies the equation

$$U^n(t,u) = e^{t\nu\Delta}u + \int_0^t e^{(t-s)\nu\Delta} B^{(n)}(U^n(s,u))\,ds.$$

Navier–Stokes solution as the limit of Galerkin-type approximations

Main result

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $l > \frac{d}{2} + \alpha + 1$. Then, there exist an H^α-valued stochastic process $U(t, \omega)$ and a subset $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure such that for all $\omega \in \Omega'$ and for all $t \in [0, T]$,

$$U(t,\omega) = e^{t\Delta\,\nu}\, U(0,\omega) - \int_0^t e^{(t-s)\Delta\,\nu} B(U(s,\omega))\, ds,$$

and the law of $U(0, \omega)$ on H^α is the measure γ.

Navier–Stokes solution as the limit of Galerkin-type approximations

Outline of Proof

Navier–Stokes solution as the limit of Galerkin-type approximations

Outline of Proof

- Let ν^n be the law of U^n on $\mathrm{C}([0,T],H^\alpha)$,

$$\nu^n(G) = \gamma\{u \in H^\alpha : U^n(\cdot,u) \in G\}, \quad G \subset \mathrm{C}([0,T],H^\alpha), \text{ Borel subset}$$

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Navier–Stokes solution as the limit of Galerkin-type approximations

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- We use the tightness criterium for probability measures ($\{\nu_n\}$ is tight), Prokhorov's theorem ($\{\nu_n\}$ is sequentially compact)
- By Skorohod's theorem, we conclude that there are H^α-valued stochastic processes $U'^n(t,\omega)$ and $U(t,\omega)$, $\omega \in \Omega$, with the laws ν^n and ν on $\mathrm{C}([0,T],H^\alpha)$, such that $U'^n(\cdot,\omega) \to U(\cdot,\omega)$ $\mathbb{P}$-a.s.

Navier–Stokes solution as the limit of Galerkin-type approximations

Outline of Proof and Corollary

Navier–Stokes solution as the limit of Galerkin-type approximations

Outline of Proof and Corollary

- $U'^n(t,\omega) = e^{t\Delta\,\nu}\,U'^n(0,\omega) + \int_0^t e^{(t-s)\Delta\,\nu} B^{(n)}(U'^n(s,\omega))\,ds \quad \mathbb{P}\text{-a.s.}$

Navier–Stokes solution as the limit of Galerkin-type approximations

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Navier–Stokes solution as the limit of Galerkin-type approximations

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Corollary

Let $l > \frac{d}{2} + \alpha + 2$. Then, there exists a subset $\Omega'' \subset \Omega$ of full $\mathbb{P}$-measure so that the stochastic process $U(t,\omega)$ constructed in Theorem on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ verifies the equation

$$U(t,\omega) = U(0,\omega) + \int_0^t (\nu\Delta U(s,\omega) - B(U(s,\omega))\, ds$$

for all $t \in [0, T]$ and for all $\omega \in \Omega''$.

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