Spatial Asymptotic Behaviour for the Boussinesq Flow in Half Space

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The Boussinesq Equations in $\mathbb{R}^n$

Heat convection in a viscous incompressible fluid under the influence of gravity is described through the Boussinesq Equations:

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= g\theta \quad \text{in} \quad \mathbb{R}^n \times [0, T), \\ \theta_t - \Delta\theta + (u \cdot \nabla)\theta &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0, T), \\ \operatorname{div} u &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0, T), \\ u(0) &= u_0 \quad \text{in} \quad \mathbb{R}^n, \\ \theta(0) &= \theta_0 \quad \text{in} \quad \mathbb{R}^n. \end{aligned}$$

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In our case we assume the gravity g belongs to $L^\infty_{n-1}(\mathbb{R}^n)^n$, i.e.

$$\|g\|_{L^\infty_{n-1}} := \operatorname{ess\,sup}(1 + |x|)^{n-1}|g(x)| < \infty.$$

A mild solution solves the integral equations:

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(u \cdot \nabla u)ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(g\theta)(s)ds,$$

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Theorem 1: (Existence and Uniqueness of Mild Solutions)
Let $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ *such that* $\operatorname{div} u_0 = 0, \mu \in (0, n], \nu > \max\{0,\ \mu - n + 1\}$ *and* $g \in L^\infty_{n-1}(\mathbb{R}^n)^n$*.*
Then there exist $T > 0$ *and a unique mild solution*

$$(u, \theta) \in C_\omega((0, T]; L^\infty_\mu(\mathbb{R}^n)^n) \times C_\omega((0, T]; L^\infty_\nu(\mathbb{R}^n)).$$

In particular, we can choose T *such that*

$$\|u_0\|_{L^\infty_\mu} + \|\theta_0\|_{L^\infty_\nu} + \|g\|_{L^\infty_{n-1}} \le \frac{C}{\sqrt{T}(1+\sqrt{T})}.$$

Sketch of Proof:

- Using the representation of the Helmholtz projection:

$$\mathbb{P} = \left(\delta_{j,h} + \mathcal{R}_j \mathcal{R}_h\right)_{j,h=1}^{n}.$$

With this we can apply the following well-known result of the kernel $e^{t\Delta}\mathbb{P}$:

The operator $O_{j,h,t} := \Delta^{-1}\partial_j\partial_h e^{t\Delta}$ is a convolution operator, where the kernel satisfies $K_{j,h,t}(x) = t^{-\frac{n}{2}} K_{j,h}\left(\frac{x}{\sqrt{t}}\right)$ for a smooth function $K_{j,h}$ such that

$$(1 + |x|)^{n+|\alpha|}\, \partial^\alpha K_{j,h} \in L^\infty(\mathbb{R}^n) \qquad \textit{for all } \alpha \in \mathbb{N}^n.$$

- Proving the boundedness of the operators

$$\mathcal{B} : C_\omega\left([0,T];L^\infty_\mu\right) \times C_\omega\left([0,T];L^\infty_\nu\right) \to C_\omega\left([0,T];L^\infty_\mu\right),$$
$$\mathcal{B}\left(u,\theta\right)(t) := -\int_0^t e^{-(t-s)\Delta}\mathbb{P}\left(u\cdot\nabla u\right)(s)ds,$$

$$\mathcal{C} : C_\omega\left([0,T];L^\infty_\nu\right) \to C_\omega\left([0,T];L^\infty_\mu\right),$$
$$\mathcal{C}\left(\theta\right)(t) := \int_0^t e^{-(t-s)\Delta}\mathbb{P}\left(g\theta\right)(s)ds,$$

$$\mathcal{D} : C_\omega\left([0,T];L^\infty_\mu\right) \times C_\omega\left([0,T];L^\infty_\nu\right) \to C_\omega\left([0,T];L^\infty_\nu\right),$$
$$\mathcal{D}\left(u,\theta\right)(t) := -\int_0^t e^{-(t-s)\Delta}u\cdot\nabla\theta(s)ds.$$

- Verifying the existence of a fixed point of the equation

$$u(t) = e^{t\Delta} u_0 + \mathcal{B}(u, \theta)(t) + \mathcal{C}(\theta)(t),$$
$$\theta(t) = e^{t\Delta} \theta_0 + \mathcal{D}(u, \theta)(t)$$

  in the Banach space $C_\omega\left([0, T]; L^\infty_\mu\right) \times C_\omega\left([0, T]; L^\infty_\nu\right)$ by using the Kato iteration method.

By smallness on the initial data we can get an arbitrary large interval of time for the existence of solutions.

Theorem 2: (Strong Solvability) *Let* $g \in W^{1,\infty}_{n-1}$, $u_0 \in L^\infty_\mu$ *with* $\operatorname{div} u_0 = 0$, $\mu \in (0, n]$, *and* $\theta_0 \in L^\infty_\nu$ *such that* $\nu > \max\{0, \ \mu - n + 1\}$. *Then the mild solution* (u, θ) *given by Theorem 1 satisfies*

$$u, \theta \in C^1\left((0, T]; \mathrm{BUC}\right) \ \cap \ C\left((0, T]; W^{2,\infty}\right)$$

and

$$\partial_t u - \Delta u + \mathbb{P}(u \cdot \nabla u) = \mathbb{P}(g\theta),$$
$$\partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0.$$

Sketch of Proof:

- By a fixed point argument we can show that

$$t^{\frac{1}{2}}\partial_i u \in C_\omega\left([0,T]; L^\infty_\mu\right),$$
$$t^{\frac{1}{2}}\partial_i \theta \in C_\omega\left([0,T]; L^\infty_\nu\right).$$

Using the inclusion $W^{1,\infty} \subseteq \mathrm{BUC}$ yields

$$u, \theta \in C\left([\varepsilon, T]; \mathrm{BUC}\right).$$

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- Furthermore, due to the embedding $W^{1,\infty}\setminus\mathbb{R} \subseteq \dot{B}^s_{\infty,1}$, for all $s\in(0,1)$, and the boundedness of the Helmholtz projection $\mathbb{P}$ on these Besov spaces we obtain by an interation

$$u,\theta \in C\left((0,T];\dot{B}^s_{\infty,1}\right) \quad \text{for all } s\in(0,3).$$

- Finally, we see $C\left((0,T];W^{2,\infty}\right)$.

Spatial Asymptotics

Theorem 3: (Spatial Asymptotic Behaviour)
Let $\kappa > 0$. For $\mu > \frac{n+2}{2}, \nu \geq 3$ and an initial data $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$, let (u, θ) be the solution of the preceding theorems. The following profile holds for $|x| >> \sqrt{t}$:

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$$
\begin{aligned}
u(x,t) = {} & e^{t\Delta} u_0(x) - \nabla \left[\gamma_{n,2} \sum_{h=1}^{n} \frac{x_h}{|x|^n} \cdot \int_0^t \int_{\mathbb{R}^n} (g_h \theta) dyds \right] \\
& - \nabla \left[\gamma_{n,1} \sum_{h,k=1}^{n} \left(\frac{x_h x_k}{|x|^{n+2}} - \frac{\delta_{h,k}}{n|x|^n} \right) \cdot \int_0^t \int_{\mathbb{R}^n} (u_h u_k + y_k g_h \theta) dyds \right] \\
& + \mathcal{O}_t(|x|^{-n-2+\kappa}), \\
\theta(x,t) = {} & e^{t\Delta} \theta_0(x) + \mathcal{O}_t(|x|^{-\mu-\nu}).
\end{aligned}
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For illustration we look at the term

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}(g\theta)ds$$

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- We rewrite $e^{t\Delta}\mathbb{P}$ as the kernel of a convolution operator:

$$E_{j,h}(x,t) := \gamma_{n,2}\partial_j \frac{x_h}{|x|^n} + |x|^{-n}\Psi\left(\frac{x}{\sqrt{t}}\right).$$

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- Seperation of Variables: Define v_h such that

$$(g_h\theta)(x,t) = \mathcal{G}(x)\int_{\mathbb{R}^n}(g_h\theta)(y,t)dy + v_h(x,t).$$

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- We estimate the remainder terms.

Boussinesq Equations in $\mathbb{R}^n_+$

Now we consider the Boussinesq Equations in the half space $\mathbb{R}^n_+$:

$$\begin{aligned}
u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= g\theta \quad \text{in} \quad \mathbb{R}^n_+ \times [0, T), \\
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\operatorname{div} u &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \times [0, T), \\
u(0) = u_0, \quad \theta(0) &= 0 \quad \text{in} \quad \mathbb{R}^n_+, \\
u(x,t) = 0, \quad \theta(x,t) &= 0 \quad \text{on} \quad \partial\mathbb{R}^n_+ \times [0, T).
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In this case we assume the gravity g being vertical, i.e.

$$g_j \equiv 0 \quad \text{for all } j \neq n$$

and belongs to $L^\infty_{0,n-1}(\mathbb{R}^n_+)^n$, i.e.

$$\|g\|_{L^\infty_{0,n-1}} := \operatorname{ess\,sup}(1 + |x_n|)^{n-1}|g(x)| < \infty.$$

Some Difficulties:

- The Helmhotz projection is not bounded in $L^\infty_\mu(^n_+)$.
- The Stokes operator $A_+ := -\mathbb{P}_+\Delta$ does not generate an strongly continuous semigroup in $L^\infty_\mu(\mathbb{R}^n_+)$.
- In contrast to the whole space, the semigroup e^{tA_+} is not just a convolution and is not bounded in $L^\infty_\mu(\mathbb{R}^n_+)$.

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Theorem 4: (Strong L^q-Solvability)

Let $n < p, q < \infty$. Assume $u_0 \in L^p_\sigma(\mathbb{R}^n_+)^n$, $\theta_0 \in L^q(\mathbb{R}^n_+)$ and $g \in L^\infty_{0,n-1}(\mathbb{R}^n_+) \times W^{1,r}(\mathbb{R}^n_+)$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then there exists $T > 0$ such that the Boussinesq Equations have a unique strong solution (u, θ) on $[0, T]$, which possesses the following properties:

$$u \in C\left((0, T]; W^{2,p}(\mathbb{R}^n_+)^n\right) \times C^1\left((0, T]; L^p_\sigma(\mathbb{R}^n_+)^n\right),$$
$$\theta \in C\left((0, T]; W^{2,q}(\mathbb{R}^n_+)\right) \times C^1\left((0, T]; L^q(\mathbb{R}^n_+)\right).$$

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Back to L^∞_μ-Setting. Let $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n_+) \times L^\infty_{n-1+\nu,\nu}(\mathbb{R}^n_+)$ with $\mu > 0$ and $\nu > \mu - n + 1$ we get a strong $L^p \times L^q$-solution (u, θ) of the Boussinesq Equation for suitable $p, q > 0$.

In the case of strong solvability we will use the Green´s Matrix

$$(\mathbb{G}_{i,j})_{i,j=1}^n : \qquad \mathrm{PartI} + \mathrm{PartII} := \mathbb{G}_{i,j} := \delta_{i,j} \left[\mathcal{G}_t(x-y) - \mathcal{G}_t(x-y^*) \right] + 4(1-\delta_{j,n})\partial_j \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_i \mathcal{L}(x-z) \mathcal{G}_t(z-y^*) dz$$

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$$\begin{aligned} u_i(x,t) = & \sum_{j=1}^n \int_{\mathbb{R}^n_+} \mathbb{G}_{i,j}(x,y,t) u_{0,j}(y) dy \\ & + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^n_+} \mathbb{G}_{i,j}(x,y,t-\tau)\left[\mathbb{P}(u\cdot\nabla u)\right]_j dy d\tau \\ & + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^n_+} \mathbb{G}_{i,j}(x,y,t-\tau)\left[\mathbb{P}(g\theta)\right]_j dy d\tau \end{aligned}$$

where we notate by $\mathcal{L}$ and $\mathcal{G}$ the fundamental solutions of the Laplace and the Heat Equations, respectively; And further $x^* := (x', x_n)$.

With this representation formula we can show the following result:
Theorem 5: *Let* $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n_+)^n \times L^\infty_{\nu+n-1;\ \nu}(\mathbb{R}^n_+)$ *such that* div $u_0 = 0, \mu \in (0, n-1], \nu > \max\{0,\ \mu - n + 1\}$. *Furthermore the assumptions of the* L^p*-Solvability are satisfied, where* $p > \max\{n, \frac{n}{\mu}\}$, $q > \max\{n, \frac{n}{\nu}\}$.
Then the given solution (u, θ) *also possesses*

$$(u, \theta) \in C((0, T]; L^\infty_\mu(\mathbb{R}^n_+)^n) \times C((0, T]; L^\infty_\nu(\mathbb{R}^n_+)).$$

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In general we can not expect that the initial data inherits the decay behaviour during the evolution. In the whole space case we already proved an optimal deacy of $|x|^{-n}$. *The following asymptotic profiles constitute a similar effect for the Boussinesq Flow in the half space.*

Spatial Asymptotics in $\mathbb{R}^n_+$

Defining an extension operator E on vector valued functions which extends the first $n-1$ components even and the n-th component odd to the last variable:

$$(Eu)'(x', x_n) := \begin{cases} u'(x', x_n) & , x_n \geq 0 \\ u'(x', -x_n) & , x_n < 0, \end{cases}$$

$$(Eu)_n(x', x_n) := \begin{cases} u_n(x', x_n) & , x_n \geq 0 \\ -u_n(x', -x_n) & , x_n < 0. \end{cases}$$

Furthermore, we extend the temperature odd:

$$\theta_E(x', x_n) := \begin{cases} \theta_n(x', x_n) & , x_n \geq 0 \\ -\theta_n(x', -x_n) & , x_n < 0. \end{cases}$$

With this we can rewrite the solution θ as the restriction on the half space of the solution

$$\theta_E(x,t) = e^{t\Delta}\theta_{0,E}(x) - \int_0^t e^{(t-s)\Delta} Eu \cdot \nabla\theta_E(x,s)ds$$

of the whole space. The heat equation achieves the homogenous boundary condition by such a reflection principle, unfortunately the Navier-Stokes Equations do not.

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Furthermore, we can define the Helmholtz projection on the half space through

$$\mathbb{P}_+ := r\mathbb{P}E.$$

PartI

Let f be an integrable function. Then we obtain by the Fourier transform $\mathcal{F}$:

$$\mathcal{F}\left(\int_{\mathbb{R}^n_+}\left[\mathcal{G}_t(x-y)-\mathcal{G}_t(x-y^*)\right]f(y)dy\right)$$
$$=\begin{cases}\mathcal{F}\left(\mathcal{G}_t\right)\cdot\mathcal{F}\left(f\right) & \text{, if } f \text{ is odd}\\ \mathcal{F}\left(\mathcal{G}_t\right)\cdot\mathcal{F}\left(\operatorname{sgn}(x_n)f\right) & \text{, if } f \text{ is even}\end{cases}.$$

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We will assume that $\mu > n$ and $\nu > 1$. It suffices to concentrate on the term $g\theta$ and replacing f with

$$\left[\delta_{j,n} + \mathcal{R}_j \mathcal{R}_n\right] \left(Eg\theta\right)_n .$$

Due to the symmetry of the introduced extension operator E we get:

$\mathrm{E(g}\theta)_n$ is odd and $\begin{cases} \mathcal{R}_j\mathcal{R}_n\left(Eg\theta\right)_n & \text{is even if } j \neq n, \\ \mathcal{R}_n\mathcal{R}_n\left(Eg\theta\right)_n & \text{is odd} \end{cases}.$

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PartII

Let f be an integrable and bounded function. Then we obtain by the Fourier transform $\mathcal{F}$:

$$\begin{aligned} &\mathcal{F}\left[\chi_{(0,\infty)}(x_n)\int_{\mathbb{R}^n_+}\partial_j\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_i\mathcal{L}(x-z)\mathcal{G}_t(z-y^*)dz\cdot f(y)dy\right] \\ &= \ldots \\ &= \frac{1}{2}\mathcal{F}\left(\mathcal{R}_j\mathcal{R}_i\mathcal{G}_t\right)\cdot\left(\mathcal{F}(f)-\mathcal{F}(\mathrm{sgn}(x_n)f)\right)+\mathcal{O}_t(e^{-x_n^2/16t}). \end{aligned}$$

We have to replace f with $(Eu_0)_j$ and $[\delta_{j,n}+\mathcal{R}_j\mathcal{R}_n](Eg\theta)_n$, $j\neq n$.

Theorem 6: (Spatial Asymptotic Behaviour in $\mathbb{R}^n_+$)
Let $\delta > 0$, $\mu > n$, $\nu > 1$ and $0 < \varepsilon < \nu - 1$. Furthermore, an initial data $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n_+)^n \times L^\infty_{\nu+n-1,\ \nu}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ is given and the assumptions of the L^p-Solvability are satisfied. Then the following profile holds for $|x| >> \sqrt{t}$, $|x_n| \geq \delta|x'|$, $i = 1, ..., n$:

Theorem 6: (Spatial Asymptotic Behaviour in $\mathbb{R}^n_+$)
Let $\delta > 0$, $\mu > n$, $\nu > 1$ and $0 < \varepsilon < \nu - 1$. Furthermore, an initial data $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n_+)^n \times L^\infty_{\nu+n-1,\ \nu}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ is given and the assumptions of the L^p-Solvability are satisfied. Then the following profile holds for $|x| >> \sqrt{t}$, $|x_n| \geq \delta|x'|$, $i = 1, ..., n$:

$$u_i(x,t) = -\gamma \sum_{j=1}^{n-1} \left(n(n+2)\frac{x_i x_j^2 x_n}{|x|^{n+4}} - n\frac{(2-\delta_{i,n})x_i x_n + 2\sigma_{i,j}(x)x_j}{|x|^{n+2}} \right) \times \int_0^t \int_{\mathbb{R}^n_+} \mathcal{T}(g_n\theta)(y,s)dyds \quad + \mathcal{O}_t\left(|x|^{-n-1}\right),$$

where $\sigma_{i,j}(x) := \delta_{i,j}x_n + \delta_{i,n}x_j$ and $\gamma := \frac{\pi^{-1/2}}{2}\Gamma\left(\frac{n}{2}\right)$.

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Thank you for your attention!