Pointwise multipliers and diffeomorphisms in function spaces

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The problem setting

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where f is an element of a function space (Besov, Triebel-Lizorkin type) and φ is a suitably smooth function.

The aim:

If φ fulfils ..., then P_φ resp. D_φ maps the function space A into A.

The spaces C^k

Let C^k be the space of all k-times differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f|C^k\| := \sum_{|\alpha|\leq k} \sup |D^\alpha f(x)| < \infty.$$

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Then

$$f, g \in C^k \Rightarrow f \cdot g \in C^k \text{ and } \|f \cdot g|C^k\| \leq c_k \|f|C^k\| \cdot \|g|C^k\|$$

and

$$(\forall f \in C^k : f \cdot g \in C^k) \Rightarrow g \in C^k \text{ and } \|P_g : C^k \to C^k\| \geq \|g|C^k\|.$$

Proof: Leibniz rule and $1 \in C^k$.

The Hölder spaces $\mathcal{C}^k$

Let $0 < \sigma \leq 1$ and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. We define

$$\|f|lip^\sigma\| := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\sigma},$$

Let $s > 0$ and $s = \lfloor s \rfloor + \{s\}$ with $\lfloor s \rfloor \in \mathbb{Z}$ and $\{s\} \in (0,1]$. Then the Hölder space with index s is given by

$$\mathcal{C}^s = \left\{ f \in C^{\lfloor s \rfloor} : \|f|\mathcal{C}^s\| := \|f|C^{\lfloor s \rfloor^-}\| + \sum_{|\alpha|=\lfloor s \rfloor} \|D^\alpha f|lip^{\{s\}}\| < \infty \right\}.$$

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It holds

$$f, g \in \mathcal{C}^s \Rightarrow f \cdot g \in \mathcal{C}^s \text{ and } \|f\cdot g|\mathcal{C}^s\| \leq c_s \|f|\mathcal{C}^s\| \cdot \|g|\mathcal{C}^s\|.$$

and

$$(\forall f \in \mathcal{C}^s : f \cdot g \in \mathcal{C}^s) \Rightarrow g \in \mathcal{C}^s \text{ and } \|P_g : \mathcal{C}^s \to \mathcal{C}^s\| \geq \|g|\mathcal{C}^s\|$$

Proof: Leibniz rule for Hölder spaces and $1 \in \mathcal{C}^s$.

The Lebesgue spaces L_p

Let $0 < p \leq \infty$ and L_p the usual set of equivalence classes of measurable functions f with finite

$$\|f|L_p\| := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}} & , 0 < p < \infty \\ ess\sup |f(x)| & , p = \infty \end{cases}$$

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Then

$$f \in L_p, g \in L_\infty \Rightarrow f \cdot g \in L_p \text{ and } \|f \cdot g|L_p\| \leq \|f|L_p\| \cdot \|f|L_\infty\|$$

and

$$(\forall f \in L_p : f \cdot g \in L_p) \Rightarrow g \in L_\infty \text{ and } \|P_g : L_p \to L_p\| \geq \|g|L_\infty\|$$

The Sobolev spaces W_p^k (i)

Let $1 < p < \infty$, $k \in \mathbb{N}_0$ and W_p^k the set of equivalence classes of measurable functions f with finite

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$$\|f|W_p^k\| := \sum_{|\alpha|\leq k} \|D^\alpha f(x)|L_p\|.$$

Then

$$f \in W_p^k, g \in C^k \Rightarrow f \cdot g \in W_p^k \text{ and } \|f \cdot g|W_p^k\| \leq \|f|W_p^k\| \cdot \|f|C^k\|.$$

The converse is not true!

The Sobolev spaces W_p^k (ii)

Theorem (Sobolev embedding)

Let $k_1 < k_2$ *and* $k_1 - \frac{n}{p_1} \leq k_2 - \frac{n}{p_2}$. *Then*

$$W_{p_2}^{k_2} \hookrightarrow W_{p_1}^{k_1}.$$

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Theorem (Multiplier algebra)

If $k > \frac{n}{p}$, *then*

$$\|f \cdot g | W_p^k\| \leq \|f | W_p^k\| \cdot \|g | W_p^k\|.$$

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Theorem (Multiplier algebra)

If $k > \frac{n}{p}$, *then*

$$\|f \cdot g | W_p^k\| \leq \|f | W_p^k\| \cdot \|g | W_p^k\|.$$

Proof: We start with

$$\|D^\alpha(f \cdot g) | L_p\| \leq c \sum \|(D^\beta f) \cdot (D^{\alpha-\beta} g) \| L_p\|$$

The Sobolev spaces W_p^k (iii)

$$
\begin{aligned}
\|D^\alpha(f\cdot g)|L_p\| &\le c\sum \|(D^\beta f)\cdot(D^{\alpha-\beta}g)\|L_p\| \\
&\le c\sum \|(D^\beta f)|L_{p_1}\|\cdot\|(D^{\alpha-\beta}g)|L_{p_2}\| \\
&\le c\sum \|f|W_{p_1}^{|\beta|}\|\cdot\|g|W_{p_2}^{|\alpha|-|\beta|}\| \\
&\le c'\|f|W_p^k\|\cdot\|g|W_p^k\|.
\end{aligned}
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The Sobolev spaces W_p^k (iii)

$$\begin{aligned} \|D^\alpha(f\cdot g)|L_p\| &\leq c\sum \|(D^\beta f)\cdot(D^{\alpha-\beta}g)|L_p\| \\ &\leq c\sum \|(D^\beta f)|L_{p_1}\|\cdot\|(D^{\alpha-\beta}g)|L_{p_2}\| \\ &\leq c\sum \|f|W_{p_1}^{|\beta|}\|\cdot\|g|W_{p_2}^{|\alpha|-|\beta|}\| \\ &\leq c'\|f|W_p^k\|\cdot\|g|W_p^k\|. \end{aligned}$$

Here $(|\alpha|\leq k)$

$$\begin{aligned} \frac{1}{p_1}+\frac{1}{p_2} &= \frac{1}{p} \\ |\beta|-\frac{n}{p_1} &\leq k-\frac{n}{p} \\ |\alpha|-|\beta|-\frac{n}{p_2} &\leq k-\frac{n}{p} \end{aligned}$$

This is possible, if $k>\frac{n}{p}$.

The Sobolev spaces W_p^k (iv)

Theorem (see e.g Runst and Sickel 1996)

The spaces $W_p^k \cap L_\infty$ are multiplier algebras, even

$$\|f \cdot g|W_p^k\| \leq c \left(\|f|W_p^k\| \cdot \|g|L_\infty\| + \|g|W_p^k\| \cdot \|f|L_\infty\| \right)$$

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Theorem (see e.g. Triebel 2008)

If W_p^k is a multiplier algebra, then φ is a pointwise multiplier for W_p^k iff

$$\sup_{m \in \mathbb{Z}} \|\psi(\cdot - m) \cdot \varphi|W_p^k\| < \infty,$$

where ψ is a nonnegative C_0^∞-function with

$$\sum_m \psi(x - m) = 1 \text{ for } x \in \mathbb{R}^n.$$

Resolution of unity

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that *supp* $\varphi_0 \subset \left\{|x| \leq \frac{3}{2}\right\}$ and $\varphi_0(x) = 1$ for $|x| \leq 1$. We define

$$\varphi(x) := \varphi_0(x) - \varphi_0(2x) \text{ and } \varphi_j(x) := \varphi(2^{-j}x) \text{ for } j \in \mathbb{N}.$$

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Then we have

$$\begin{aligned} \sum_{j=0}^{\infty} \varphi_j(x) &= 1. \\ |D^\alpha \varphi_j(x)| &\leq c_\alpha 2^{-j|\alpha|}, \\ \textit{supp } \varphi_j &\subset \left\{2^{j-1} \leq |x| \leq 2^{j+1}\right\}, \end{aligned} \tag{1}$$

A sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ with (1), $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and φ_0 as above will be called resolution of unity.

The definition of $B^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee}|L_p\|^q\right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B^{s,\varphi}_{p,q}(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi} < \infty\right\}.$$

Then $(B^{s,\varphi}_{p,q}(\mathbb{R}^n), \|\cdot|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi})$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent norms. So we denote it shortly by $B^s_{p,q}(\mathbb{R}^n)$.

The definition of $F^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|F^s_{p,q}(\mathbb{R}^n)\|^\varphi := \left\| \left(\sum_{j=0}^\infty 2^{jsq} |(\varphi_j \hat{f})^\vee|^q \right)^{\frac{1}{q}} \middle| L_p \right\|$$

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Atomic characterization of $B^s_{p,q}(\mathbb{R}^n)$

Theorem

Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $K, L \geq 0$, $K > s$ and $L > \sigma_p - s$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \quad \textit{with convergence in } \mathcal{S}'(\mathbb{R}^n).$$

Here $a_{\nu,m}$ are $(s,p)_{K,L}$-atoms located at $Q_{\nu,m}$ and $\|\lambda|b_{p,q}\| < \infty$. Furthermore, we have in the sense of equivalence of norms

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \sim \inf \ \|\lambda|b_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f.

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Treatment of products using atomic decompositions

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If $\varphi \cdot a_{\nu,m}$ are atoms: $\varphi \cdot f \in A^s_{p,q}(\mathbb{R}^n)$

The definition of atoms

A function $a : \mathbb{R}^n \to \mathbb{R}$ is called classical $(s,p)_{K,L}$-atom located at $Q_{\nu,m}$ if

$$supp\ a \subset d \cdot Q_{\nu,m}$$

$$|D^\alpha a(x)| \leq C \cdot 2^{-\nu\left(s-\frac{n}{p}\right)+|\alpha|\nu} \text{ for all } |\alpha| < K+1, \qquad (2)$$

$$\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 \text{ for all } |\beta| < L. \qquad (3)$$

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A function $a : \mathbb{R}^n \to \mathbb{R}$ is called $(s,p)_{K,L}$-atom located at $Q_{\nu,m}$ if instead of (2) and (3) it holds (for all $\psi \in \mathcal{C}^L$)

$$\|a(2^{-\nu}\cdot)|\mathcal{C}^K\| \leq C \cdot 2^{-\nu(s-\frac{n}{p})}$$

$$\left|\int_{d\cdot Q_{\nu,m}} \psi(x)a(x)\ dx\right| \leq C \cdot 2^{-\nu\left(s+L+n\left(1-\frac{1}{p}\right)\right)} \|\psi|\mathcal{C}^L\|$$

Atomic representations revisited

Every classical $(s,p)_{K,L}$-atom is an $(s,p)_{K,L}$-atom.

Theorem

The atomic representation theorem for $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ is valid with both forms of atoms. Hence every f which can be represented as a linear combination of classical $(s,p)_{K,L}$-atom resp. $(s,p)_{K,L}$-atom belongs to $B^s_{p,q}(\mathbb{R}^n)$ resp. $F^s_{p,q}(\mathbb{R}^n)$. Hereby

$$K > s \quad \text{and}$$

$$L > \sigma_p - s = \sigma_p = n\left(\frac{1}{p} - 1\right)_+ - s \quad \text{resp.}$$

$$L > \sigma_{p,q} - s = n\left(\frac{1}{\min(p,q)} - 1\right)_+ - s$$

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The proof for classical atoms goes back to Triebel '97. The modifications were suggested by Skrzypczak '98, Triebel/Winkelvoss '96.

The pointwise multiplier theorem (i)

Now we get

Lemma

There exists a constant c with the following property: For all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}$, *all* $(s,p)_{K,L}$*-atoms* $a_{\nu,m}$ *with support in* $d \cdot Q_{\nu,m}$ *and all* $\varphi \in C^\rho$ *with* $\rho \geq \max(K,L)$ *the product*

$$c \cdot \left\|\varphi|\mathcal{C}^\rho\right\|^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is an $(s,p)_{K,L}$*-atom with support in* $d \cdot Q_{\nu,m}$.

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$$c \cdot \|\varphi|\mathcal{C}^\rho\|^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is an $(s,p)_{K,L}$-atom with support in $d \cdot Q_{\nu,m}$.

Proof: Use that $\mathcal{C}^\rho$ is a multiplication algebra.

This does not work for classical atoms $(s,p)_{K,L}$-atoms with $L \geq 1$, since in general moment conditions are destroyed when multiplying by φ!

The pointwise multiplier theorem (ii)

We get as a Corollary

Theorem

Let $s \in \mathbb{R}$ *and* $0 < q \leq \infty$.
(i) Let $0 < p \leq \infty$ *and* $\rho > \max(s, \sigma_p - s)$. *Then there exists a positive number c such that*

$$\|\varphi f | B^s_{p,q}(\mathbb{R}^n)\| \leq c \|\varphi | \mathcal{C}^\rho\| \cdot \|f | B^s_{p,q}(\mathbb{R}^n)\|$$

for all $\varphi \in \mathcal{C}^\rho$ *and all* $f \in B^s_{p,q}(\mathbb{R}^n)$.
(ii) Let $0 < p < \infty$ *and* $\rho > \max(s, \sigma_{p,q} - s)$. *Then there exists a positive number c such that*

$$\|\varphi f | F^s_{p,q}(\mathbb{R}^n)\| \leq c \|\varphi | \mathcal{C}^\rho\| \cdot \|f | F^s_{p,q}(\mathbb{R}^n)\|$$

for all $\varphi \in \mathcal{C}^\rho$ *and all* $f \in F^s_{p,q}(\mathbb{R}^n)$.

The diffeomorphism theorem (i)

In the same way we can treat the mapping D_φ:

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \;\Rightarrow\; f \circ \varphi = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot (a_{\nu,m} \circ \varphi).$$

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Hence we have to investigate if $a_{\nu,m} \circ \varphi$ is an $(s,p)_{K,L}$-atom when $a_{\nu,m}$ is an $(s,p)_{K,L}$-atom.

The diffeomorphism theorem (ii)

Definition

Let $\rho \geq 1$.

(i) Let $\rho = 1$. We say that the map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a 1-diffeomorphism if φ is a bi-Lipschitzian map, i.e. that there are constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq c_2.$$

for all $x, y \in \mathbb{R}^n$ with $0 < |x - y| \leq 1$.

(ii) Let $\rho > 1$. We say that the one-to-one map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a ρ-diffeomorphism if the n components φ_i of φ have classical derivatives up to order $\lfloor \rho \rfloor$ with $\frac{\partial \varphi_i}{\partial x_j} \in \mathcal{C}^{\rho-1}$ and $|\det J(\varphi)(x)| \geq c > 0$ for all $x \in \mathbb{R}^n$.

The diffeomorphism theorem (iii)

Theorem

(i) Let $0 < p \le \infty$, $\rho \ge 1$ *and* $\rho > \max(s, 1 + \sigma_p - s)$. *If* φ *is a* ρ*-diffeomorphism, then there exists a constant* c *such that*

$$\|f(\varphi(\cdot))|B^s_{p,q}(\mathbb{R}^n)\| \le c\|f|B^s_{p,q}(\mathbb{R}^n)\|.$$

for all $f \in B^s_{p,q}(\mathbb{R}^n)$. *Hence* D_φ *maps* $B^s_{p,q}(\mathbb{R}^n)$ *onto* $B^s_{p,q}(\mathbb{R}^n)$.
(ii) Let $0 < p < \infty$, $\rho \ge 1$ *and* $\rho > \max(s, 1 + \sigma_{p,q} - s)$. *If* φ *is a* ρ*-diffeomorphism, then there exists a constant* c *such that*

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(ii) Let $0 < p < \infty$, $\rho \geq 1$ *and* $\rho > \max(s, 1 + \sigma_{p,q} - s)$. *If* φ *is a* ρ*-diffeomorphism, then there exists a constant* c *such that*

$$\|f(\varphi(\cdot))|F^s_{p,q}(\mathbb{R}^n)\| \leq c\|f|F^s_{p,q}(\mathbb{R}^n)\|.$$

for all $f \in F^s_{p,q}(\mathbb{R}^n)$. *Hence* D_φ *maps* $F^s_{p,q}(\mathbb{R}^n)$ *onto* $F^s_{p,q}(\mathbb{R}^n)$.

Proof: Show that $a_{\nu,m}$ is an $(s,p)_{K,L}$-atom, control the support of the atoms and verify convergence

The end

Thank you for your attention

Questions?