# Hardy spaces with variable exponents and generalized Campanato spaces

Yoshihiro Sawano (Kyoto Univ.)
yosihiro@math.kyoto-u.ac.jp
and Eiichi Nakai (Ibaraki Univ.)

2011 September Germany Tabarz

This work is based upon the paper

Eiichi Nakai and Yoshihiro Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, submitted.

# Motivation

Unified understanding of complicated aspect of Hardy spaces

Recall that the Hardy space is defined by the norm given by

$$\|f\|_{H^p} = \left\| \sup_{t>0} t^{-n} |\psi(t^{-1}\cdot) * f| \right\|_p$$

for $0 < p < \infty$ and $f \in \mathcal{S}'$, where $\psi \in \mathcal{S}$ has nonzero integral.

1. The Hardy space $H^p$ with $1 < p < \infty$ coincides with $L^p$.
2. The Hardy space $H^1$ is a proper subset of $L^1$.
3. The Hardy space $H^p$ with $0 < p < 1$ is not contained in $L^1_{\mathrm{loc}}$.

Therefore, the theory of Hardy spaces is very complicated.

To mix $L^{p_0}$ and $L^{p_1}$ with $p_0 \neq p_1$. In particular let us consider $0 < p_0 < 1 < p_1 < \infty$. Then how do we mix them ?

1. Use variable Lebesgue spaces.

2. Use Orlicz spaces.

In this talk, we take up both of them.

# 2. Variable Lebesgue spaces and Orlicz spaces

**2.1. Variable Lebesgue spaces** Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function such that $0 < \inf_{x \in \mathbb{R}^n} p(x) \leqq \sup_{x \in \mathbb{R}^n} p(x) < \infty$.

The space $L^{p(\cdot)}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $p(\cdot)$, is defined as the set of all measurable functions $f$ for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)} \, dx$ is finite for some $\varepsilon > 0$. We let

$$\|f\|_{L^{p(\cdot)}} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leqq 1 \right\}$$

for such $f$. As a special case of the theory of Nakano and Luxemberg, we see that $(L^{p(\cdot)}, \| \cdot \|_{L^{p(\cdot)}})$ is a quasi-normed space.

The function $p(\cdot)$ is called the variable exponent. It is customary to denote $p_+ \equiv \sup_{x \in \mathbb{R}^n} p(x)$ and $p_- \equiv \inf_{x \in \mathbb{R}^n} p(x)$, which we shall do throughout this talk. As is often the case with many other cases, we postulate on $p(\cdot)$ the following conditions.

(log-Hölder continuity)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \leqq \frac{1}{2}, \tag{1}$$

(decay condition)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \text{for} \quad |y| \geqq |x|. \tag{2}$$

Denote by $p_\infty$ the limit $\lim_{x \to \infty} p(x)$ ensured by the decay condition.

We have the following impotant control of the maximal operator. The (Hardy-Littlewood) maximal operator $M$ is given by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, dy.$$

The following result is from the paper below:

> D. Cruz-Uribe, SFO, A. Fiorenza, J. M. Martell and C. Pérez, The boundedness of classical operators on variable $L^p$ spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.

Lemma Suppose that $p(\cdot) : \mathbb{R}^n \to [0, \infty)$ satisfies the log-Hölder continuity and the decay condition. Let $1 < p_- \leqq p_+ < \infty$ and let $u \in (1, \infty]$. Then we have $\left\| \left( \sum_{j=1}^{\infty} M{f_j}^u \right)^{\frac{1}{u}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^{p(\cdot)}}$ for all sequences of measurable functions $\{f_j\}_{j=1}^{\infty}$.

Lemma Let $Q$ be a cube.

1. If $|Q| \leqq 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p_-(Q)}$.
2. If $|Q| \geqq 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p_\infty}$.

**Outline of the proof.**

1. This follows from the log-Hölder continuity.

$$|p(x) - p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \leqq \frac{1}{2}.$$

2. This follows from the decay condition.

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \text{for} \quad |y| \geqq |x|.$$

**2.2. Orlicz spaces** The class of Orlicz spaces is wider than the class of Lebesgue spaces and Orlicz spaces are generated by functions $\Phi : [0, \infty) \to [0, \infty)$, the condition of which we now describe. Let $\Phi^*$ be the set of all functions $\Phi : [0, \infty) \to [0, \infty)$ such that $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and that $\lim_{r \to \infty} \Phi(r) = \infty$. For $\Phi \in \Phi^*$, we define Orlicz spaces $L^\Phi(\mathbb{R}^n)$ (in the wide sense) as the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\|f\|_{L^\Phi} < \infty$, where the quantity $\|f\|_{L^\Phi}$ is given by

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left( \frac{|f(x)|}{\lambda} \right) dx \leqq 1 \right\}.$$

If $\Phi(r) = r^p$ $(0 < p < \infty)$, then $L^\Phi(\mathbb{R}^n)$ is isomorphic to the usual $L^p(\mathbb{R}^n)$ space with norm coincidence. If $\Phi$ is convex, then the functional $\| \cdot \|_{L^\Phi}$ is a norm and thereby $L^\Phi(\mathbb{R}^n)$ is a Banach space. In the present talk we are interested in a more general subclass of $\Phi^*$.

Now we clarify and restrict the class of $\Phi$.

1. For functions $\Phi$ and $\Psi$ in $\Phi^*$, write $\Phi \approx \Psi$ if there exists a constant $C \geqq 1$ such that

$$\Phi\left(\frac{r}{C}\right) \leqq \Psi(r) \leqq \Phi(Cr) \quad \text{for all} \quad r \geqq 0.$$

   In this case one say that $\Phi$ is equivalent to $\Psi$.

2. Let $\Phi \in \Phi^*$ and $\ell \in (0,1]$. One says that $\Phi$ is $\ell$-convex if $\Phi((\cdot)^{1/\ell})$ is convex. One also says that $\Phi$ is quasi $\ell$-convex if $\Phi \approx \Psi$ for some $\ell$-convex function $\Psi \in \Phi^*$.

3. Let $\Phi \in \Phi^*$ and $\ell \in [1,\infty)$. One says that $\Phi$ is $\ell$-concave if $\Phi((\cdot)^{1/\ell})$ is concave. One also says that $\Phi$ is quasi $\ell$-concave if $\Phi \approx \Psi$ for some $\ell$-convex function $\Psi \in \Phi^*$.

## Remarks

($A$) It turned out that the failure of rearrangement invariance in variable Lebesgue space made things much more difficult. We overcame this difficulty by using the lemma in the next page.

($B$) We do not know the condition of $\Phi$ is good enough or not.

The next lemma is crucial for our observations in the variable exponent setting.

Lemma There exists $\delta > 0$ with the following properties. Suppose we are given a countable collection of nonnegative numbers $\{\kappa_j\}_{j=1}^{\infty}$ and non-zero measurable functions $\{f_j\}_{j=1}^{\infty}$ such that $f_j \in L^1(Q_j)$ for each $j \in \mathbb{N}$. Then, we have

$$\left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\kappa_j |f_j|^{\delta} |Q_j|^{\delta}}{\|f_j\|_{L^1(Q_j)}^{\delta} \|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{\min(1,p_-)} \right\}^{\frac{1}{\min(1,p_-)}} \right\|_{L^{p(\cdot)}}$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\kappa_j |\chi_{Q_j}|^{\delta} |Q_j|^{\delta}}{\|\chi_{Q_j}\|_{L^1(Q_j)}^{\delta} \|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{\min(1,p_-)} \right\}^{\frac{1}{\min(1,p_-)}} \right\|_{L^{p(\cdot)}}$$

# 3. Hardy spaces with variable exponent and Orlicz Hardy spaces

Definition

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of norms $\{p_N\}_{N \in \mathbb{N}}$ given by $p_N(\varphi) \equiv \sum_{|\alpha| \leqq N} \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$ for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \leqq 1\}$.
2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{M}f$ the grand maximal operator given by $\mathcal{M}f(x) \equiv \sup\{|t^n \psi(t\cdot) * f(x)| : t > 0, \quad \psi \in \mathcal{F}_N\}$, where we choose and fix a large integer $N$.
3. The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H^{p(\cdot)}} \equiv \|\mathcal{M}f\|_{L^{p(\cdot)}}$ is finite.

Do the same for Orlicz spaces to obtain Hardy-Orlicz spaces.

Below we concentrate on Hardy spaces with variable exponent, since they are somewhat parallel.

We have

1. Poisson integral characterization.

2. Equivalent norm, for example,

$$\|f\|_{L^{p(\cdot)}} \sim \left\| \sup_{t>0} t^{-n} |\psi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}$$

   whenever $\psi \in \mathcal{S}(\mathbb{R}^n)$ has nonzero integral.

3. $\mathcal{S}(\mathbb{R}^n) \subset H^{p(\cdot)}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

# 3.1. Atomic decomposition

**Definition**

A function $a$ is said to be a $(p(\cdot), q)$-atom if it is supported on a cube $Q$ with the following properties.

1. $\|a\|_q \leqq \dfrac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$

2. $\displaystyle\int_Q a(x)x^\alpha \, dx = 0$ for all $\alpha$ with $\|\alpha\|_1 \leqq n\left(\frac{1}{p_-} - 1\right)$.

The following is an important observation in our talk.

Suppose that we are given cubes $\{Q_j\}_{j=1}^\infty$ and complex constants $\{\lambda_j\}_{j=1}^\infty$. Then define

$$\|\{\lambda_j\}_{j=1}^\infty\|_{\mathcal{A}_1} := \left\| \left( \sum_{j=1}^\infty \frac{|\lambda_j|^{1}\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}{}^{1}} \right)^{1/1} \right\|_{L^{p(\cdot)}}$$

$$\|\{\lambda_j\}_{j=1}^\infty\|_{\mathcal{A}_2} := \left\| \left( \sum_{j=1}^\infty \frac{|\lambda_j|^{\min(1,p_-)}\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}^{\min(1,p_-)}} \right)^{1/\min(1,p_-)} \right\|_{L^{p(\cdot)}}.$$

**Theorem** Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then TFAE.

1. $f \in H^{p(\cdot)}(\mathbb{R}^n)$.
2. There exists $(p(\cdot), \infty)$-atom such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with

$$\|\{\lambda_j\}_{j=1}^{\infty}\|_{\mathcal{A}_1} = \left\| \sum_{j=1}^{\infty} \frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}} < \infty.$$

3. There exists $(p(\cdot), q)$-atom such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with

$$\|\{\lambda_j\}_{j=1}^{\infty}\|_{\mathcal{A}_2} = \left\| \left( \sum_{j=1}^{\infty} \frac{|\lambda_j|^{\min(1,p_-)} \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}^{\min(1,p_-)}} \right)^{1/\min(1,p_-)} \right\|_{L^{p(\cdot)}} < \infty.$$

Here $q$ in the definition of atoms satisfies $q \gg 1$.

## Remark

1. We did not calculate how large $q$ should be in the above theorem.
2. This theorem holds whenever $0 < p_- \leqq p_+ < \infty$. This is what we said; we could unify the theory of Hardy spaces.
3. Simultaneous convergence: If $f \in H^{p_1(\cdot)}(\mathbb{R}^n) \cap H^{p_2(\cdot)}(\mathbb{R}^n)$, then
$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$
converges in the topology of $H^{p_1(\cdot)}(\mathbb{R}^n) \cap H^{p_2(\cdot)}(\mathbb{R}^n)$.
4. This function spaces turns out to cover the space due to Diening, Hästo and Roudenko. Indeed, we have the following theorem:

Theorem If $\psi \in \mathcal{S}(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} \psi(x)\,dx \neq 0$, and if one defines the local hardy norm $\|\cdot\|_{h^{p(\cdot)}}$ by

$$\|f\|_{h^{p(\cdot)}} = \left\| \sup_{0<t<1} \sup_{\varphi \in \mathcal{F}_N} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}, \tag{3}$$

then we have the following norm equivalence:

$$\begin{aligned} \|f\|_{h^{p(\cdot)}} &\sim \|f\|_{F^0_{p(\cdot)2}} \\ &\sim \|(1-\psi(D))f\|_{H^{p(\cdot)}} + \|\psi(D)f\|_{L^{p(\cdot)}} \\ &\sim \|(1-\psi(D))f\|_{\dot{F}^0_{p(\cdot)2}} + \|\psi(D)f\|_{L^{p(\cdot)}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \end{aligned}$$

## Other corollaries

1. Molecular decomposition

2. Characterization by means of the local means

# 4. Dual spaces

## 4.1. Campanato spaces

In this section we assume that

$$0 < p_- \leqq p_+ \leqq 1.$$

Defintion

Let $L \geqq n(1/p_- - 1)_+$ and $\phi : \{\text{cubes}\} \to (0, \infty)$ be a function. Let $f \in L^q_{\text{loc}}$. Denote by $P^L_Q f$ the unique polynomial of order $L$ $P$ such that

$$\int_Q f(x)Q(x)dx = \int_Q P^L_Q f(x)Q(x)dx$$

for all polynomials $Q(X)$ of order $L$. Then define

$$\|f\|_{\mathcal{L}_{\phi,q,L}} = \sup_{Q\,:\,cube} \frac{1}{\phi(Q)} \left( \frac{1}{|Q|} \int_Q |f(x) - P^L_Q f(x)|^q \, dx \right)^{1/q}.$$

## Duality

Let $p(\cdot)$ be as usual and define $\phi(Q) = \|\chi_Q\|_{L^{p(\cdot)}}$. Then for all $g \in \mathcal{L}_{\phi,q,L}(\mathbb{R}^n)$ there exists a bounded linear functional $L_g$ on $H^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^d} g(x)a(x)dx = L_g(a)$$

for all $(p(\cdot), \infty)$-atoms $a$. And all bounded linear functional in $H^{p(\cdot)}(\mathbb{R}^n)$ is realized in this way.

## Remarks

1. Using $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$-duality, we can easily construct $g$ starting from a functional $L$.

2. However, the things are not so easy because, if we prove such that

$$\int_{\mathbb{R}^d} g(x)a(x)dx = L_g(a)$$

for all $(p(\cdot),\infty)$-atoms $a$, a passage to $f \in H^{p(\cdot)}(\mathbb{R}^n)$ in general is quite difficult. We use Simulateneous convergenece to overcome this difficulty.

# 4.2 Littlewood-Paley characterization of dual space $\mathcal{L}_{\phi,q,L}(\mathbb{R}^n)$.

As usual, we define $\Delta_h^k$ to be a difference operator, which is defined inductively by

$$\Delta_h^1 f = \Delta_h f \equiv f(\cdot + h) - f, \quad \Delta_h^k \equiv \Delta_h^1 \circ \Delta_h^{k-1}, \quad k \geqq 2. \tag{4}$$

Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and $d \in \mathbb{N} \cup \{0\}$. Then $\Lambda_{\phi,d}(\mathbb{R}^n)$, the Hölder space with variable exponent $p(\cdot)$, is defined to be the set of all continuous functions $f$ such that $\|f\|_{\Lambda_{\phi,d}} < \infty$, where

$$\|f\|_{\Lambda_{\phi,d}} \equiv \sup_{x \in \mathbb{R}^n,\, h \neq 0} \frac{1}{\phi(x,|h|)} \left| \Delta_h^{d+1} f(x) \right|.$$

One considers elements in $\Lambda_{\phi,d}(\mathbb{R}^n)$ modulo polynomials of degree $d$ so that $\Lambda_{\phi,d}(\mathbb{R}^n)$ is a Banach space. When one writes $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$, then $f$ stands for the representative of $\{f + P :$ $P$ is a polynomial of degree $d$ $\}$.

Below we present a typical theorem:

**Theorem** Assume that $\phi : \{cubes\} \to (0, \infty)$ satisfies the following conditions.

(A1) There exists a constant $C > 0$ such that

$$C^{-1} \leqq \frac{\phi(x,r)}{\phi(x,2r)} \leqq C, \quad (x \in \mathbb{R}^n,\ r > 0).$$

(A2) There exists a constant $C > 0$ such that

$$C^{-1} \leqq \frac{\phi(x,r)}{\phi(y,r)} \leqq C, \quad (x, y \in \mathbb{R}^n,\ r > 0,\ |x-y| \leqq r).$$

(A3) There exists a constant $C > 0$ such that

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \leqq C\phi(x,r), \quad (x \in \mathbb{R}^n,\ r > 0).$$

Then the function spaces $\Lambda_{\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ are isomorphic. Speaking more precisely, we have the following :

1. For any $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ we have $\|f\|_{\mathcal{L}_{q,\phi,d}} \lesssim \|f\|_{\Lambda_{\phi,d}}$.
2. Any element in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ has a continuous representative. Moreover, whenever $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ and we have $\|f\|_{\Lambda_{\phi,d}} \lesssim \|f\|_{\mathcal{L}_{q,\phi,d}}$.

We remark that as a special case of $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ we can realize the dual of $H^{p(\cdot)}(\mathbb{R}^n)$.

## Open problems

How do we characterize the dual of $H^{p(\cdot)}(\mathbb{R}^n)$ for general cases, that is, without assuming $p_+ \leqq 1$ ?

Let us describe how difficult it is.

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to B^{n/p-n}_{\infty\infty}(\mathbb{R}^n) \text{ for } p < 1$$

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to \mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty 2}(\mathbb{R}^n) \text{ for } p = 1$$

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to F^0_{p'2}(\mathbb{R}^n) \text{ for } p > 1$$

and we encounter different types of function spaces.

**Thank you for your attention!**