Inverse problem for the p-Laplacian

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Calderón problem

Medical imaging, Electrical Impedance Tomography:

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ bounded domain, γ positive function.

Boundary measurements given by Dirichlet-to-Neumann map (DN map)

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine γ.

Calderón problem

Different questions:

1. Boundary uniqueness: if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.
2. Interior uniqueness: if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.
3. Reconstruction: algorithm for recovering γ from Λ_γ.
4. Stability: if Λ_{γ_1} and Λ_{γ_2} are close, then γ_1 and γ_2 are close.
5. Partial data: if $\Lambda_{\gamma_1}|_\Gamma = \Lambda_{\gamma_2}|_\Gamma$ for $\Gamma \subseteq \partial\Omega$, then $\gamma_1 = \gamma_2$.

Calderón problem

A number of positive results are known:

1. Boundary uniqueness: Kohn-Vogelius (1984), Sylvester-Uhlmann (1988), Brown (2001)
2. Interior uniqueness: Sylvester-Uhlmann (1987), Astala-Päivärinta (2006)
3. Reconstruction: Nachman (1988)
4. Stability: Alessandrini (1988)
5. Partial data: Kenig-Sjöstrand-Uhlmann (2007)

Nonlinear equations

Similar inverse problems for nonlinear models from physics:

- elasticity
- fluid dynamics
- Einstein equations

Nonlinear models also of mathematical interest.

Nonlinear equations

Dirichlet problem for nonlinear conductivity $a = a(x, u, q)$,

$$\begin{cases} \mathrm{div}(a(x, u, \nabla u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ bounded domain, a positive function satisfying suitable conditions.

Boundary measurements given by nonlinear DN map

$$\Lambda_a : f \mapsto a(x, u, \nabla u)\nabla u \cdot \nu|_{\partial\Omega}.$$

(Note that $a \mapsto \Lambda_a$ is nonlinear even if the equation is linear.)

Inverse problem: given Λ_a, determine a.

Linearization

Let $\gamma = \gamma(x, u) \in C^2(\overline{\Omega} \times \mathbb{R})$. If $z \in \mathbb{R}$ and $f \in C^{2,\alpha}(\partial\Omega)$, then

$$\lim_{t \to 0} \frac{\Lambda_\gamma(z + tf) - \Lambda_\gamma(z)}{t} = \Lambda_{\gamma^z}(f)$$

in $W^{1-1/p,p}(\partial\Omega)$. Here $\gamma^z(x) = \gamma(x, z)$.

Theorem

(Sun 1996) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ for two conductivities $\gamma_j = \gamma_j(x, u)$, then $\gamma_1 \equiv \gamma_2$.

Proof.

By linearization $\Lambda_{\gamma_1^z} = \Lambda_{\gamma_2^z}$ for all $z \in \mathbb{R}$, and the interior uniqueness result in the linear case shows that $\gamma_1 = \gamma_2$. □

Results for nonlinear equations

Nonlinearity depending on x and u:

- Isakov (1993)
- Isakov-Sylvester (1994), Isakov-Nachman (1995)
- Sun (1996), Sun-Uhlmann (1997)

Nonlinearities with certain dependence on ∇u:

- Isakov (2001), Hervas-Sun and Kang-Nakamura (2002)

Kang-Nakamura:

$$a(x, \nabla u)\nabla u = \gamma(x)\nabla u + \sum_{j,k=1}^{n} c_{jk}(x)\partial_j u \partial_k u + R(x, \nabla u)$$

p-Laplacian

Nonlinear model based on the *p-Laplace operator*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty.$$

The corresponding p-Laplace equation $\Delta_p u = 0$, whose solutions are called *p-harmonic functions*, is a prototypical nonlinear equation in divergence form.

Euler-Lagrange equation for minimizing the p-Dirichlet energy

$$E_p(u) = \int_\Omega |\nabla u|^p \, dx$$

over $u \in W^{1,p}(\Omega)$ with fixed boundary values. Applications in fluid dynamics and image processing.

p-Laplacian

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, bounded C^1 domain, $1 < p < \infty$, and $\gamma \in L^\infty(\Omega)$ positive. Dirichlet problem for $u \in W^{1,p}(\Omega)$,

$$\begin{cases} \operatorname{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Nonlinear DN map

$$\Lambda_\gamma : \; B_{pp}^{1-1/p}(\partial\Omega) \to B_{p'p'}^{-1+1/p}(\partial\Omega),$$
$$f \mapsto \gamma(x)|\nabla u|^{p-2}\nabla u \cdot \nu|_{\partial\Omega}$$

where u is the unique solution with boundary values f.

Main results

Denote by $\Lambda_\gamma^{\mathbb{R}}$ and $\Lambda_\gamma^{\mathbb{C}}$ the DN maps acting on real and complex boundary values.

Theorem

(S-Zhong) If $\gamma_1, \gamma_2 \in C(\overline{\Omega})$ and $\Lambda_{\gamma_1}^{\mathbb{R}} = \Lambda_{\gamma_2}^{\mathbb{R}}$, then $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.

Theorem

(S-Zhong) If $\gamma_1, \gamma_2 \in C(\overline{\Omega})$ and $\Lambda_{\gamma_1}^{\mathbb{C}} = \Lambda_{\gamma_2}^{\mathbb{C}}$, then $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.

Proofs are constructive and local: construct explicit functions $f_M \in C^1(\partial\Omega)$, supported near a boundary point x_0, so that

$$\lim_{M\to\infty} \int_{\partial\Omega} \Lambda_\gamma(f_M)\bar{f}_M \, dS = \gamma(x_0).$$

Linearization

Consider the linearization of Λ_γ at constant boundary value $z \in \mathbb{R}$.

The solution of $\operatorname{div}(\gamma|\nabla u|^{p-2}\nabla u) = 0$ with boundary value $z + tf$ is $z + tu_f$, where u_f is the solution for boundary data f. Then

$$\begin{aligned}\Lambda_\gamma(z+tf) &= \gamma|\nabla(z+tu_f)|^{p-2}\nabla(z+tu_f)\cdot\nu|_{\partial\Omega}\\ &= t^{p-1}\Lambda_\gamma(f).\end{aligned}$$

The linearization does not even exist if $p < 2$, and would not give new information anyway.

Method

For linear conductivity equation ($p = 2$), results are based on harmonic exponentials $e^{\rho \cdot x}$, $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$.

In the nonlinear case $p \neq 2$, we construct complex geometrical optics (or Faddeev type) solutions near the boundary, built upon p-harmonic complex exponentials $e^{\rho \cdot x}$. Here $\rho \cdot \rho \neq 0$.

In the case of real valued boundary measurements we cannot use complex exponentials. However, these can be replaced by certain real p-harmonic functions introduced by Wolff.

Complex p-harmonic exponentials

Lemma

Let $h(x) = e^{\rho \cdot x}$, $\rho = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}^n$. Then $\Delta_p h = 0$ if and only if $(p-1)|\alpha|^2 = |\beta|^2$ and $\alpha \cdot \beta = 0$.

Proof.

Since $\nabla h = \rho e^{\rho \cdot x}$, we have

$$\begin{aligned}
\mathrm{div}(|\nabla h|^{p-2}\nabla h) &= \mathrm{div}(|\rho|^{p-2} e^{(p-2)\alpha\cdot x} \rho e^{\rho\cdot x}) \\
&= \mathrm{div}(|\rho|^{p-2} \rho e^{(p-1)\alpha\cdot x + i\beta\cdot x}) \\
&= |\rho|^{p-2}\rho \cdot ((p-1)\alpha + i\beta) e^{(p-1)\alpha\cdot x + i\beta \cdot x}.
\end{aligned}$$

Here $\rho \cdot ((p-1)\alpha + i\beta) = (p-1)|\alpha|^2 - |\beta|^2 + ip\alpha \cdot \beta$. □

Real valued case

In the real valued case, isolate the properties of exponentials used in the proof and look for corresponding real p-harmonic functions.

Lemma

(Wolff 1984) Let $h(x) = e^{-x_n} a(x_1)$. One has $\Delta_p h = 0$ in $\mathbb{R}^n_+$ if

$$a''(x_1) + V(a, a')a = 0,$$

where

$$V(a, a') = \frac{(2p-3)(a')^2 + (p-1)a^2}{(p-1)(a')^2 + a^2}.$$

Any solution a is smooth and periodic with period $\lambda = \lambda(p)$, and satisfies $\int_0^\lambda a(x_1)\, dx_1 = 0$.

Tools

- Well-posedness for Dirichlet problem in $W^{1,p}(\Omega)$
- Inequalities for $\xi, \eta \in \mathbb{R}^n$ (typically $\xi = \nabla u$):

$$|\eta|^p \geq |\xi|^p + p|\xi|^{p-2}\xi \cdot (\eta - \xi),$$
$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \lesssim (|\xi| + |\eta|)^{p-2}|\xi - \eta|,$$
$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \sim (|\xi| + |\eta|)^{p-2}|\xi - \eta|^2.$$

- Hardy's inequality: if $\delta(x) = \mathrm{dist}(x, \partial\Omega)$

$$\|u/\delta\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad u \in W_0^{1,p}(\Omega).$$

Well-posedness

Proposition

Let $\gamma \in L^\infty(\Omega)$ positive. For any $f \in W^{1,p}(\Omega)$ the problem

$$\begin{cases} \operatorname{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u - f \in W_0^{1,p}(\Omega) \end{cases}$$

has a unique solution $u \in W^{1,p}(\Omega)$.

Proof.

Show that solutions are unique minimizers of

$$E(u) = \int_\Omega \gamma |\nabla u|^p \, dx$$

in admissible class $\mathcal{A} = \{u \in W^{1,p}(\Omega)\,;\, u - f \in W_0^{1,p}(\Omega)\}$.

Well-posedness

Let u be a minimizer. Then for any $\varphi \in W_0^{1,p}(\Omega)$

$$E(u + t\varphi) = \int_\Omega \gamma(|\nabla u + t\nabla\varphi|^2)^{p/2}\,dx$$
$$= \int_\Omega \gamma(|\nabla u|^2 + 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2)^{p/2}\,dx$$

and

$$0 = \frac{d}{dt}E(u + t\varphi)\Big|_{t=0} = p\int_\Omega \gamma|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx$$
$$= -p\int_\Omega \mathrm{div}(\gamma|\nabla u|^{p-2}\nabla u)\varphi\,dx.$$

Thus $\mathrm{div}(\gamma|\nabla u|^{p-2}\nabla u) = 0$ in Ω and minimizers are solutions. The converse also holds.

Well-posedness

Let $(v_j) \subseteq \mathcal{A}$ be a sequence such that $E(v_j) \to E_0$ where $E_0 = \inf_{v \in \mathcal{A}} E(v)$. Since $f \in \mathcal{A}$, we have $E_0 < \infty$ and

$$E(v_j) = \int_\Omega |\nabla v_j|^p \, dx \leq C < \infty.$$

Using that $v_j - f \in W_0^{1,p}(\Omega)$, the Poincaré inequality implies

$$\|v_j - f\|_{W^{1,p}} \leq C \|\nabla (v_j - f)\|_{L^p} \leq C < \infty.$$

By weak compactness there exists $u \in W^{1,p}(\Omega)$ with $v_j \rightharpoonup u$ and $\nabla v_j \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$, and since $W_0^{1,p}$ is closed we have $u - f \in W_0^{1,p}(\Omega)$. The function u is a minimizer:

$$\begin{aligned} E(v_j) &= \int_\Omega \gamma |\nabla v_j|^p \, dx \\ &\geq \int_\Omega \gamma (|\nabla u|^p + p|\nabla u|^{p-2} \nabla u \cdot (\nabla v_j - \nabla u)) \, dx \end{aligned}$$

and by weak convergence $E_0 = \lim_{j \to \infty} E(v_j) \geq E(u)$.

Outline of proof

Assume $x_0 = 0$ and $\partial\Omega$ is flat near 0. We convert the p-harmonic function $e^{\rho \cdot x}$ into an exact solution of $\mathrm{div}(\gamma|\nabla u|^{p-2}\nabla u) = 0$ in Ω concentrating near 0.

Define approximate solution

$$u_0(x) = \eta(Mx)h(Nx)$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\eta = 1$ for $|x| \leq 1/2$, and

$$h(x) = e^{(i\beta - e_n)\cdot x}$$

with $\beta \in \mathbb{R}^n$ satisfying $|\beta|^2 = p - 1$ and $\beta \cdot e_n = 0$. Then $\mathrm{div}(\gamma|\nabla u_0|^{p-2}\nabla u_0) \approx 0$ in Ω if $M/N = o(1)$ as $M \to \infty$.

Outline of proof

We obtain an exact solution u by solving the Dirichlet problem with boundary values u_0,

$$\begin{cases} \operatorname{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Let $f = u_0|_{\partial\Omega}$. Then we have

$$\int_{\partial\Omega} \Lambda_\gamma(f)\bar{f}\, dS = \int_\Omega \gamma|\nabla u|^{p-2}\nabla u \cdot \nabla \bar{u}_0\, dx$$
$$= \int_\Omega \gamma|\nabla u_0|^p\, dx + \int_\Omega \gamma(|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0)\cdot\nabla\bar{u}_0\, dx.$$

Since f is explicit, the left hand side is determined by Λ_γ. We recover $\gamma(0)$ by taking the limit as $M \to \infty$.

Outline of proof

$$\int_{\partial\Omega} \Lambda_\gamma(f)\bar{f}\, dS = \int_\Omega \gamma |\nabla u_0|^p\, dx + \int_\Omega \gamma(|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0) \cdot \nabla \bar{u}_0\, dx.$$

As $M \to \infty$, can compute

$$M^{n-1}N^{1-p}\int_\Omega \gamma|\nabla u_0|^p\, dx \to c_p\gamma(0).$$

Inequality $||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \lesssim (|\xi| + |\eta|)^{p-2}|\xi - \eta|$ gives

$$\left|\int_\Omega \gamma(|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0) \cdot \nabla \bar{u}_0\, dx\right| \lesssim O((M^{1-n}N^{p-1})^{\frac{p-1}{p}})\|\nabla u - \nabla u_0\|_{L^p}.$$

Outline of proof

Recall

$$\begin{cases} \operatorname{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Also

$$\int_\Omega |\nabla u_0|^p \, dx \sim M^{1-n} N^{p-1}.$$

Enough to prove

$$\int_\Omega |\nabla u - \nabla u_0|^p \, dx = o(M^{1-n} N^{p-1}).$$

Outline of proof

Lemma

$$\int_\Omega |\nabla u - \nabla u_0|^p \, dx = o(M^{1-n}N^{p-1}).$$

Proof uses the following facts:

- u_0 is explicit and supported in $B(0, 1/M)$
- $\sup_{x \in B(0,1/M)} |\gamma(x) - \gamma(0)| = o(1)$, replace γ by $\gamma(0)$
- u is a solution
- $u - u_0 \in W_0^{1,p}(\Omega)$ can be used as a test function
- inequalities for pth powers of vectors, Hardy's inequality

Questions

The results suggest a nonlinear complex geometrical optics construction for the p-Laplacian.

1. Boundary uniqueness: higher order derivatives, boundary stability?
2. Interior uniqueness?
3. Reconstruction?
4. Stability?
5. Partial data?

Other nonlinear equations?