LOCAL MEANS, WAVELET BASES/ WAVELET REPRESENTATIONS AND WAVELET ISOMORPHISMS IN BESOV-MORREY AND TRIEBEL-LIZORKIN-MORREY SPACES

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1. Morrey spaces

Definition

For $0 < u \le p \le \infty$ the spaces

$$M_{pu} := M_{pu}(\mathbb{R}^n) := \{f \in L_u^{loc}(\mathbb{R}^n) : \ \|f|M_{pu}\| < \infty\}$$

are called Morrey spaces, their quasi-norms $\|\cdot|M_{pu}\|$ for $u < \infty$ are defined as

$$\begin{aligned} \|f|M_{pu}\| &:= \sup_{R>0, x\in\mathbb{R}^n} R^{n\left(\frac{1}{p}-\frac{1}{u}\right)} \left(\int_{B_R(x)} |f(y)|^u \, dy\right)^{\frac{1}{u}} \\ &= \sup_{R>0, x\in\mathbb{R}^n} R^{n\left(\frac{1}{p}-\frac{1}{u}\right)} \|f\|_{L_u(B_R(x))} \, . \end{aligned}$$

For $u = \infty$ we define the norm $\|\cdot|M_{\infty,\infty}\|$ analogue by using the essential supremum.

- $\|f|M_{pp}\| = \|f\|_{L_p}$.
- If $p = \infty$ then $M_{\infty u} = L_\infty$ with norm equivalence.

1. Morrey spaces

- $(M_{pu}, \|\cdot|M_{pu}\|)$ is a quasi-Banach space and if $u \geq 1$ a Banach space.
- $M_{pu} \hookrightarrow M_{p'u'}$ holds if, and only if, $0 < u' \leq u \leq p' = p < \infty$ or $0 < u', u \leq p' = p = \infty$. In particular $L_p \hookrightarrow M_{pu'}$ for $0 < u' \leq p$.
- If $0 < u < p$ then $w - L_p \hookrightarrow M_{pu}$ where $\|f|w - L_p\| := \sup_{t>0} \left(t^p \lambda_f(t)\right)^{\frac{1}{p}}$ and $\lambda_f(t) := |\{x : |f(x)| > t\}|$. In particular $L_{pq} \hookrightarrow M_{pu}$ where L_{pq} denotes the Lorentz spaces and $L_{p\infty} = w - L_p$.
- $L_\infty \cap M_{pu}$ is not dense in M_{pu} for $u < p$. In particular, $S(\mathbb{R}^n)$ is not dense in M_{pu} for $u < p$.
- Let $w_\beta(y) := \left(1 + |y|^2\right)^{\frac{\beta}{2}}$ for $\beta \in \mathbb{R}$ and $\|f|L_u(w_\beta)\| := \left(\int_{\mathbb{R}^n} |f(y)|^u w_\beta(y) dy\right)^{\frac{1}{u}}$. For $0 < u \leq p$ there is a weight $w := w_\beta$ such that $M_{pu} \hookrightarrow L_u(w_\beta)$.

2. Besov-Triebel-Lizorkin-Morrey spaces

$\varphi_0 \in S(\mathbb{R}^n)$ with $\varphi_0 \geq 0$ and

$$\varphi_0(x) = 1 \text{ if } |x| \leq 1 \text{ and } \varphi_0(x) = 0 \text{ if } |x| \geq \frac{3}{2}. \tag{1}$$

$\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) := \varphi(2^{-j}x)\ \forall j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.
$0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Besov-Morrey spaces:

$$B^s_{(pu)q} := \left\{ f \in S'(\mathbb{R}^n) : \left\| f | B^s_{(pu)q} \right\|_{\varphi} < \infty \right\},$$

where

$$\left\| f | B^s_{(pu)q} \right\|_{\varphi} := \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| F^{-1} [\varphi_j Ff] \, | M_{pu} \right\|^q \right)^{\frac{1}{q}}.$$

If $p < \infty$

$$F^s_{(pu)q} := \left\{ f \in S'(\mathbb{R}^n) : \left\| f | F^s_{(pu)q} \right\|_{\varphi} < \infty \right\},$$

where

$$\left\| f | F^s_{(pu)q} \right\|_{\varphi} := \left\| 2^{js} F^{-1} [\varphi_j Ff] (\cdot) | M_{pu}(\ell_q) \right\|.$$

If $q = \infty$ the quasi-norms get modified using ℓ_∞. $A \in \{B, F\}$.

3. Local Means

Definition

1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define dyadic cubes

$$Q_{\nu m} := 2^{-\nu} m + 2^{-\nu} \left[-\frac{1}{2}, \frac{1}{2} \right)^n .$$

2. Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$. Then, given a doubly indexed complex sequence $\lambda = (\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, set

$$\left\| \lambda | b^s_{(pu)q} \right\| := \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu s q} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \chi_{\nu,m}(\cdot) \right| M_{pu} \Bigg\|^q \right)^{\frac{1}{q}} ,$$

$$\left\| \lambda | f^s_{(pu)q} \right\| := \left\| \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu s q} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m} \chi_{\nu,m}(\cdot)|^q \right)^{\frac{1}{q}} \right| M_{pu} \Bigg\| ,$$

$a \in \{b, f\}$.

3. Local Means

Definition (kernels of local means)

$A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$, $C > 0$. Then L_∞-functions $k_{\nu m} : \mathbb{R}^n \mapsto \mathbb{C}$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called [A,B,C]-kernels (of local means) centered at $Q_{\nu m}$ if there exists all (classical) derivatives $D^\alpha k_{\nu m}$ with $|\alpha| \le A$ such that

$$|D^\alpha k_{\nu m}(x)| \le c 2^{\nu n + \nu|\alpha|} \left(1 + 2^\nu \left|x - 2^{-\nu} m\right|\right)^{-C}, \; |\alpha| \le A, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \quad (2)$$

for all $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} x^\beta k_{\nu m}(x) dx = 0, \; |\beta| < B, \; \nu \in \mathbb{N}, \; m \in \mathbb{Z}^n. \quad (3)$$

For $\nu = 0$ or $B = 0$ no moment condition (3) is required. If $A = 0$ then (2) means $k_{\nu m} \in L_\infty$ and $|k_{\nu m}(x)| \le 2^{\nu n} \left(1 + 2^\nu |x - 2^{-\nu} m|\right)^{-C}$ for all $x \in \mathbb{R}^n$.

3. Local Means

- $A^s_{(pu)q} \hookrightarrow A^s_{uq}(w) \hookrightarrow B^s_{u,\max(u,q)}(w) \hookrightarrow B^{s-\epsilon}_{u2}(w)$
- S is dense in $B^s_{pq}(w)$ for $\max(p,q) < \infty$
- We set $f \in \left(B^s_{pq}(w)\right)'$ for $\max(p,q) < \infty$ if, and only if, there exists a constant c such that $|f(\varphi)| \leq c \left\| \varphi | B^s_{pq}(w) \right\|$ for all $\varphi \in S$.

Lemma

Let $k_{\nu m}$ be kernels where $A \in \mathbb{N}_0$ with $A > \sigma_u - s$, $B \in \mathbb{N}_0$ and $C > 0$ sufficient large. Then we obtain

$$k_{\nu m} \in \left(B^{s-\epsilon}_{u2}(w)\right)'$$

for $0 < u < \infty$ and $\epsilon > 0$ such that $A > \sigma_u - s + \epsilon$. If $u = \infty$ then we have $k_{\nu m} \in B^{-s+\epsilon}_{11}$ for $\epsilon > 0$ such that $A > -s + \epsilon$.

3. Local Means

Definition (local means via dual pairings)

$f \in A^s_{(pu)q}$ where $0 < u \leq p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. $k_{\nu m}$ be kernels with $A > \sigma_u - s$ where $\sigma_u := n\left(\frac{1}{min(1,u)} - 1\right)$ and $B \in \mathbb{N}_0$. Then we call

$$k_{\nu m}(f) := \langle f, k_{\nu m} \rangle, \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n,$$

local means, considered as a continuous extension of the dual pairing within $\langle S, S' \rangle$ (respectively the quasi-norm of $B^{s-\epsilon}_{u2}(w)$. If $p = \infty$ we define the local means through

$$k_{\nu m}(f) := \langle k_{\nu m}, f \rangle, \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n,$$

again considered as dual pairing within $\langle S, S' \rangle$ (respectively the norm of $B^{-s+\epsilon}_{1,1}$). Furthermore,

$$k(f) := \{ k_{\nu m}(f) | \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \}.$$

3. Local Means

Theorem

$0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. $k_{\nu m}$ be kernels with $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ where

$$A > \sigma_{uq} - s, B > s, \sigma_{uq} := n\left(\frac{1}{min(1, u, q)} - 1\right),$$

and $C > 0$ sufficient large are fixed. Then

$$\left\| f(k) | a^s_{(pu),q} \right\| \leq c \left\| f | A^s_{(pu),q} \right\|.$$

Remark

Up to now, for the Morrey type function spaces were just considered local means based on the special kernels

$$k_{\nu m}(y) = (\triangle^N k)(2^\nu y - m),\ \nu \in \mathbb{N},\ m \in \mathbb{Z}^n,\ y \in \mathbb{R}^n,$$

where $k \in C^\infty$ compactly supported, see the work of Sawano/ Tanaka 2007 and Sawano 2009.

4. Wavelets

Recall that a wavelet is a function $\psi(x) \in L_2(\mathbb{R})$ such that the family of functions $\left\{2^{\frac{\nu}{2}}\psi(2^\nu x - m)\right\}_{\nu\in\mathbb{Z},m\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{R})$. If several functions built up the basis by dilatation and translation we call them wavelets. If one wins the wavelets with the help of a multiresolution analysis one gets the orthonormal basis through $\left\{\psi_F(x-m), 2^{\frac{\nu}{2}}\psi_M(2^\nu x - m)\right\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}}$ where ψ_F is called scaling funtion (or father wavelet) and ψ_M the associated wavelet (or mother wavelet). We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the a tensor product procedure. Let

$$G := (G_1, \ldots, G_n) \in G^0 := \{F, M\}^n ,$$

which means that G_r is either F or M. We put

$$G^\nu := G^0 \setminus (F, \ldots, F),\ \nu \in \mathbb{N}.$$

Hence G^0 has 2^n elements, whereas G^ν with $\nu \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\psi^\nu_{G,m} := 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r),\ G \in G^\nu,\ m \in \mathbb{Z}^n,$$

where $\nu \in \mathbb{N}_0$ and ψ_F, ψ_M have the above meaning.

Then

$$\{\psi_{Gm}^{\nu} | \, \nu \in \mathbb{N}_0, \; G \in G^{\nu}, \; m \in \mathbb{Z}^n\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$, therefore

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{G \in G^{\nu}} \sum_{m \in \mathbb{Z}^n} \langle f, \psi_{Gm}^{\nu} \rangle \, \psi_{Gm}^{\nu}$$

in $L_2(\mathbb{R}^n)$. One may ask whether this orthonormal basis in $L_2(\mathbb{R}^n)$ remains to be an (unconditional) basis in other spaces on $\mathbb{R}^n$.

Definition

Let B be a quasi-Banach space and J a countable index set. A sequence $\{b_j\}_{j \in J} \subset B$ is called a (Schauder) basis if any $b \in B$ can be uniquely represented as $b = \sum_{j \in J} \lambda_j b_j$, $\lambda_j \in \mathbb{C}$ with convergence in B. The basis is called unconditional if for any rearrangement σ, $\left\{b_{\sigma(j)}\right\}_j$ is again a basis with $\sum_{j \in J} \lambda_j b_j = \sum_{j \in J} \lambda_{\sigma(j)} b_{\sigma(j)}$.

Definition

Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. *Then* $b^s_{(pu)q}$ *is the collection of all sequences*

$$\lambda = \{ \lambda^\nu_{Gm} \in \mathbb{C} | \, \nu \in \mathbb{N}_0, \; G \in G^\nu, \; m \in \mathbb{Z}^n \}$$

such that

$$\left\| \lambda | b^s_{(pu)q} \right\| := \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu sq} \sum_{G \in G^\nu} \left\| \sum_{m \in \mathbb{Z}^n} \lambda^\nu_{Gm} \chi_{\nu,m}(\cdot) \right| M_{pu} \right\|^q \right)^{\frac{1}{q}} < \infty$$

and $f^s_{(pu)q}$ *for* $p < \infty$ *is the collection of all sequences* λ *such that*

$$\left\| \lambda | f^s_{(pu)q} \right\| := \left\| \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu sq} \sum_{G \in G^\nu} \sum_{m \in \mathbb{Z}^n} |\lambda^\nu_{Gm} \chi_{\nu,m}(\cdot)|^q \right)^{\frac{1}{q}} \right| M_{pu} \right\| < \infty.$$

$a \in \{b, f\}$.

$0 < u \le p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, an orthonormal basis $\{\psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n}$ in $L_2(\mathbb{R}^n)$ which satisfies that $\{2^{+\nu\frac{n}{2}}\psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n}$ are [K,K,M]-kernels with

$$K > \max(s, \sigma_{uq} - s), \sigma_{uq} := n\left(\frac{1}{\min(1,u,q)} - 1\right),$$

and $M > 0$ sufficient large. Let $\lambda \in a^s_{(pu)q}$. Then the following assertions hold.

- The sum

$$\sum_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \psi^\nu_{Gm} \tag{4}$$

  convergences unconditionally in S' to some distribution f.

- $f \in A^s_{(pu)q}(\mathbb{R}^n)$ and

$$\left\| f|\, A^s_{(pu)q} \right\| \le c \left\| \lambda | a^s_{(pu)q} \right\|,$$

  where the constant c does not depend on λ.

- The sum (4) converges unconditionally in $B^\sigma_{u\bar{q}}\left(w^{1+\epsilon}\right)$ for any $\sigma < s$, $\epsilon > 0$, $0 < \bar{q} \le \infty$. If $u = p < \infty$ we have unconditional convergence in $A^\sigma_{(pp)\bar{q}}(\mathbb{R}^n)$ for any $\sigma < s$, $0 < \bar{q} \le \infty$. If $u = p$ and $\max(p,q) < \infty$ the sum (4) converges unconditionally in $A^s_{(pu)q}(\mathbb{R}^n)$.

Let $f \in A^s_{(pu)q}(\mathbb{R}^n)$. Then we may define the sequence λ by

$$\lambda^\nu_{Gm} = 2^{\nu \frac{n}{2}} \left\langle f, \psi^\nu_{Gm} \right\rangle, \ \nu \in \mathbb{N}_0, G \in G^\nu, m \in \mathbb{Z}^n$$

(modification if $p = \infty$) and it holds

- $\lambda \in a^s_{(pu)q}$ and
$$\left\| \lambda | a^s_{(pu)q} \right\| \leq c \left\| f | \, A^s_{(pu)q} \right\|,$$
where the constant c does not depend on f.
- The sum (4) converges to f in S' (and in the other convergencies from 14).
- This representation of f is unique, i.e. if for some sequence $\beta \in a^s_{(pu)q}$ the sum $\sum_{\nu \in \mathbb{N}_0, G \in G^\nu, m \in \mathbb{Z}^n} \beta^\nu_{Gm} 2^{-\nu \frac{n}{2}} \psi^\nu_{Gm}$ converges in S' to f then $\beta = \lambda$.

The map

$$I : f \mapsto \left\{ 2^{\nu \frac{n}{2}} \left\langle f, \psi^\nu_{Gm} \right\rangle \right\}$$

is an isomorphic map of $A^s_{(pu)q}(\mathbb{R}^n)$ onto $a^s_{(pu)q}$.
$\{\psi^\nu_{Gm}\}$ is a basis in $A^s_{(pu)q}(\mathbb{R}^n)$ if, and only if, $u = p$ and $\max(p, q) < \infty$.

Proposition

We define the truncated sequences $\lambda^l = \left\{(\lambda^{\nu}_{Gm})^l\right\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n}$ *by*

$$(\lambda^{\nu}_{Gm})^l := \begin{cases} \lambda^{\nu}_{Gm} & , \text{ if } |\nu| \le l, G \in G^{\nu}, |m| \le l \\ 0 & , \text{ else} \end{cases}$$

for every $l \in \mathbb{N}$.

1. *Let* $0 < u < p$. *If* $\lambda \in a^s_{(pu)q}$ *then for arbitrary* $\sigma \le s$ *it has not to hold* $\lambda^l \rightarrow \lambda$ *in* $a^{\sigma}_{(pu)\infty}$.
2. *Whereas we recall that for* $\max(p, q) < \infty$ *and* $\lambda \in a^s_{pq}$ *(noting* $a^s_{(pp)q} =: a^s_{pq}$*), it holds* $\lambda^l \rightarrow \lambda$ *in* a^s_{pq}.
3. *Moreover, for* $p < \infty$ *and* $\lambda \in a^s_{p\infty}$ *it has not to hold* $\lambda^l \rightarrow \lambda$ *in* $a^s_{p\infty}$*, whereas the truncated sequences converge for arbitrary* $\sigma < s$ *and* $0 < q \le \infty$ *in* a^{σ}_{pq}.
4. *The remaining case* $\lambda \in b^s_{\infty q} = a^s_{(\infty\infty)q}$ *coincides with* $\lambda \in a^s_{(\infty u)q}$ *for* $0 < u < \infty$.

3.5 Wavelets

Definition

The function f on $\mathbb{R}^n$ is called r-regular, if f belongs to the space C^r, $r \in \mathbb{Z}$, $r \geq -1$ and

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq \frac{c_k}{(1+|x|)^k}$$

for each $k \in \mathbb{N}_0$ and each multiindex α with $|\alpha| \leq \max(r, 0)$ and some constant c_k. As above C^{-1} stands for the measurable functions and C^0 for the continuous functions.

Proposition

Given an arbitrary multiresolution analysis having an r-regular scaling function (or father wavelet) we construct an r-regular associated wavelet (or mother wavelet). The r-regularity of the mother wavelet yields the fact that its first r moment conditions holds. We extend these wavelets, called r-regular wavelets, from $\mathbb{R}$ to $\mathbb{R}^n$ using the tensor product procedure and observe that $\left\{2^{+\nu\frac{n}{2}}\psi^{\nu}_{Gm}\right\}$ are $[r, r+1, M]$-kernels for any $M > 0$. Lastly, we obtain the theorem for the r-regular wavelets.

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