UMD spaces and imaginary powers of operators

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8th International Conference on Function Spaces, Differential Operators and Nonlinear Analysis

Tabarz, September 19, 2011

In collaboration with:

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✎ J.J. Betancor, A.J. Castro, J. Curbelo and L. R.-M., *Characterization of UMD Banach spaces by imaginary powers of Hermite and Laguerre operators*, to appear in Complex Analysis and Operator Theory.

✎ J.J. Betancor, R. Crescimbeni, J.C. Fariña and L. R.-M., *Multipliers and imaginary powers of the Schrödinger operators characterizing UMD Banach spaces*, preprint arXiv: 1109.0429.

1. Definitions and Preliminaries

2. Laplace transform type spectral multipliers for Hermite, Laguerre and Schrödinger operators

3. Characterization of UMD Banach spaces via imaginary powers of Hermite, Laguerre and Schrödinger operators

Unconditional Martingale Difference (UMD) spaces

Let $\mathbb{B}$ a Banach space, $\Omega \subset \mathbb{R}^n$ and μ a positive measure on Ω.

For every $1 \leq p < \infty$, consider the following spaces:

$$L^p_{\mathbb{B}}(\Omega,\mu) = \left\{ f : \Omega \to \mathbb{B} : ||f||_{L^p_{\mathbb{B}}(\Omega,\mu)} = \left(\int_\Omega ||f(x)||_{\mathbb{B}}{}^p d\mu(x) \right)^{1/p} < \infty \right\}$$

$$L^p(\Omega,\mu) \otimes \mathbb{B} = \left\{ f \in L^p_{\mathbb{B}}(\Omega,\mu) : f = \sum_{j=1}^N f_j b_j, f_j \in L^p(\Omega,\mu), b_j \in \mathbb{B} \right\}$$

We write $L^p_{\mathbb{B}}(\Omega)$, when $d\mu(x) = dx$ and $L^p(\Omega,\mu)$, when $\mathbb{B} = \mathbb{C}$.

$\mathbb{B}$ is UMD when, for some (equiv., for any) $p \in (1,\infty)$, the $\mathbb{B}$-valued Hilbert transform $\mathscr{H}$ defined in the natural way on $L^p(\mathbb{R}) \otimes \mathbb{B}$ can be extended to $L^p_{\mathbb{B}}(\mathbb{R})$ as a bounded operator from $L^p_{\mathbb{B}}(\mathbb{R})$ into itself.

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Other characterizations of UMD spaces

- By means of Riesz transforms:

✎ I. Abu-Falahah and J.L. Torrea, *Hermite functions expansions versus Hermite polynomial expansions*, Glasgow Math. J. 48 (2006), 203-215.

✎ J.J. Betancor, J.C. Fariña, L. R.-M., A. Sanabria and J.L. Torrea, *Transference between Laguerre and Hermite settings*, J. Funct. Anal. 254 (2008), 826-850.

✎ E. Harboure, J.L. Torrea and B. Viviani, *Vector-valued extensions of operators related to the Ornstein-Uhlenbeck semigroup*, J. d'Analyse Math. 91 (2003), 1-29.

- By means of Littlewood-Paley g-functions:

✎ T. Hytönen, *Aspects of probabilistic Littlewood-Paley theory in Banach spaces*, Banach spaces and their applications in analysis, Walter de Gruyter, Berlin, 2007, 343-355.

✎ T. Hytönen, *Littlewood-Paley-Stein theory for semigroups in UMD spaces*, Rev. Mat. Iberoamericana 23 (2007), 973-1009.

Characterizations of UMD spaces via imaginary powers of operators

✎ T.M. McConnell, *On Fourier multiplier transformations of Banach-valued functions*, Trans. Amer. Math. Soc. 285 (1984), 739-757.

✎ S. Guerre-Delabrière, *Some remarks on complex powers of (-Δ) and UMD spaces*, Illinois J. Math. 35 (1991), 401-407.

Consider the Laplacian operator Δ on $\mathbb{R}^n$ and $\gamma \in \mathbb{R}$.

$$(-\Delta)^{i\gamma}(f) := \left(|y|^{2i\gamma}\hat{f}\right)\check{}, \quad f \in L^2(\mathbb{R}^n),$$

where $\hat{f}$ (resp., $\check{f}$) denotes the Fourier (resp., inverse Fourier) transform of f.

Theorem [T. McConnell '84; S. Guerre-Delabrière '91]

Let $\mathbb{B}$ a Banach space. The following assertions are equivalent.
(a) $\mathbb{B}$ is UMD.
(b) For every $\gamma \in \mathbb{R}$ and for some (equiv., for any) $p \in (1,\infty)$, $(-\Delta)^{i\gamma}$ defined on $L^p(\mathbb{R}) \otimes \mathbb{B}$ can be extended as a bounded operator to $L^p_{\mathbb{B}}(\mathbb{R})$.

Proposition (J.J. Betancor, R. Crescimbeni, J.C. Fariña and L. R.-M.)

The analogous result for the n-dimensional Laplacian operator.

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Our objective:

Characterize the UMD spaces via the L^p-boundedness of imaginary powers of other differential operators

- Let $\mathbb{L}$ be a positive, unbounded and self-adjoint operator on $D(\mathbb{L}) \subset L^2(\Omega)$.
- $E_{\mathbb{L}}$ the spectral measure associated to $\mathbb{L}$.
- $m \in L^\infty[0,\infty)$.

The spectral multiplier $T_m^{\mathbb{L}}$ is defined on $L^2(\Omega)$ by

$$T_m^{\mathbb{L}}(f) = \int_{[0,\infty)} m(\lambda) E_{\mathbb{L}}(d\lambda) f, \quad f \in L^2(\Omega).$$

For every $\gamma \in \mathbb{R}$, the imaginary power $\mathbb{L}^{i\gamma}$ is given by

$$\boxed{\mathbb{L}^{i\gamma}(f) = T_{m_\gamma}^{\mathbb{L}}(f)}$$

where

$$m_\gamma(\lambda) = \lambda^{i\gamma} = \lambda \int_0^\infty e^{-\lambda t} \overbrace{\frac{t^{-i\gamma}}{\Gamma(1-i\gamma)}}^{\phi_\gamma(t)} dt, \quad \lambda \in (0,\infty).$$

➢ If $T_{m_\gamma}^{\mathbb{L}}$ can be extended boundedly on $L^p(\Omega)$, then $\mathbb{L}^{i\gamma}$ can be defined on $L^p(\Omega) \otimes \mathbb{B}$ in the natural way.

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- We say that m is a function of Laplace transform type when, for some $\phi \in L^\infty(0,\infty)$,

$$m(\lambda) = \lambda \int_0^\infty e^{-\lambda t} \phi(t)\, dt, \quad \lambda \in (0,\infty).$$

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- The spectral multiplier $T_m^{\mathbb{L}}$ associated to certain operator $\mathbb{L}$ in $L^2(\Omega,\mu)$ is a Laplace transform type multiplier when m is a function of Laplace transform type.

  - $T_m^{\mathbb{L}}$ is a bounded operator from $L^2(\Omega,\mu)$ into itself.

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The Differential Operators that we consider...

- Hermite operator **H** :

$$\mathbf{H} = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2\right), \text{ on } (\mathbb{R}, dx).$$

- Laguerre operator $\mathbf{L}_\alpha$, $\alpha > -1/2$:

$$\mathbf{L}_\alpha = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 - \frac{\alpha^2 - 1/4}{x^2}\right), \text{ on } ((0,\infty), dx).$$

- Schrödinger operator $\mathscr{L}$:

$$\mathscr{L} = -\Delta + V, \text{ on } (\mathbb{R}^n, dx), n \geq 3.$$

$V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $V \geq 0$, $V \not\equiv 0$, $V \in RH_q$ for some $q \geq \frac{n}{2}$, that is,

$$\frac{1}{|B|}\int_B [V(x)]^q dx \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right)^q, \text{ for each ball } B \subset \mathbb{R}^n.$$

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...and the corresponding spectral multipliers.

Let $m \in L^\infty(0,\infty)$.

- Spectral multiplier $T_m^{\mathbf{H}}$ associated to the Hermite operator:

$$T_m^{\mathbf{H}}(f) = \sum_{k=0}^{\infty} m(\overbrace{k+1/2}^{\lambda_k})c_k(f)h_k, \quad f \in L^2(\mathbb{R}).$$

  - ➛ h_k the k-th normalized Hermite function, $\mathbf{H}h_k = \lambda_k h_k$.
  - ➛ $c_k(f) = \int_{\mathbb{R}} f(x)h_k(x)dx$.

- Spectral multiplier $T_m^{\mathbf{L}_\alpha}$ associated to the Laguerre operator:

$$T_m^{\mathbf{L}_\alpha}(f) = \sum_{k=0}^{\infty} m(\overbrace{2k+\alpha+1}^{\lambda_k^\alpha})c_k^\alpha(f)\varphi_k^\alpha, \quad f \in L^2(0,\infty).$$

  - ➛ φ_k^α: the k-th normalized Laguerre function of type α, $\mathbf{L}_\alpha\varphi_k^\alpha = \lambda_k^\alpha h_k$.
  - ➛ $c_k^\alpha(f) = \int_0^\infty f(x)\varphi_k^\alpha(x)dx$.

- Spectral multiplier $T_m^{\mathscr{L}}$ associated to the Schrödinger operator:

$$T_m^{\mathscr{L}}(f) = \int_{[0,\infty)} m(\lambda)E_{\mathscr{L}}(d\lambda)f, \quad f \in L^2(\mathbb{R}^n).$$

Spectral multipliers as Principal Value operators

Denote $(\mathbb{L},\Omega)$=$(\mathbf{H},\mathbb{R}),(\mathbf{L}_\alpha,(0,\infty))$ or $(\mathscr{L},\mathbb{R}^n)$, $n\geq 3$.

Theorem 1

Let m a function of Laplace transform type associated to $\phi\in L^\infty(0,\infty)$. Then, there exists $\Lambda(=\Lambda_\phi)\in L^\infty(0,\infty)$ such that, for every $f\in C_c^\infty(\Omega)$,

$$T_m^{\mathbb{L}}(f)(x)=\lim_{\varepsilon\to 0^+}\left(\Lambda(\varepsilon)f(x)+\int_{\Omega,|x-y|>\varepsilon}K_\phi^{\mathbb{L}}(x,y)f(y)dy\right),\quad \text{a.e. } x\in\Omega,$$

where

$$K_\phi^{\mathbb{L}}(x,y)=-\int_0^\infty\phi(t)\frac{\partial}{\partial t}W_t^{\mathbb{L}}(x,y)dt,\quad x,y\in\Omega.$$

Moreover, if there exists the limit: $\lim_{t\to 0^+}\phi(t)=\phi(0^+)$, then

$$T_m^{\mathbb{L}}(f)(x)=\phi(0^+)f(x)+\lim_{\varepsilon\to 0^+}\int_{\Omega,|x-y|>\varepsilon}K_\phi^{\mathbb{L}}(x,y)f(y)dy,\quad \text{a.e. } x\in\Omega.$$

- $W_t^{\mathbb{L}}(x,y)$, $x,y\in\Omega$: kernel related to the heat semigroup $\{W_t^{\mathbb{L}}=e^{-t\mathbb{L}}\}_{t>0}$.

Main steps in the proof of Theorem 1: Hermite multiplier

Let m a function of Laplace transform type related to $\phi \in L^\infty(0,\infty)$, and denote by M the function

$$M(\lambda) := \lambda \int_0^\infty e^{-\lambda t} e^{-1/2t} \phi(t) dt, \ \lambda > 0.$$

- Ornstein-Uhlenbeck operator: $\mathbf{O} = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx}$

$$\mathbf{O} H_k = k H_k, k \in \mathbb{N}, \quad H_k : \ k\text{-th normalized Hermite polynomial.}$$

➢ $T_M^{\mathbf{O}}$ verifies Theorem 1:

✎ J. García-Cuerva, G. Mauceri, P. Sjögren and J.L. Torrea, *Spectral multipliers for the Ornstein-Uhlenbeck semigroup*, J. Anal. Math. 78 (1999), 401-407.

➢ $T_m^H(f)(x) = e^{-x^2/2}\left(T_M^{\mathbf{O}}(e^{y^2/2}f)(x) + A_\phi(e^{y^2/2}f)(x)\right)$, a.e. $x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$.

➢ A_ϕ is bounded on $L^p(\mathbb{R}, e^{-x^2}dx)$, $p \in (1,\infty)$, and from $L^1(\mathbb{R}, e^{-x^2}dx)$ into $L^{1,\infty}(\mathbb{R}, e^{-x^2}dx)$.

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- Ornstein-Uhlenbeck operator: $\mathbf{O} = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx}$

$$\mathbf{O}H_k = kH_k, k \in \mathbb{N}, \quad H_k : \text{ } k\text{-th normalized Hermite polynomial.}$$

➢ $T_M^{\mathbf{O}}$ verifies Theorem 1:

✎ J. García-Cuerva, G. Mauceri, P. Sjögren and J.L. Torrea, *Spectral multipliers for the Ornstein-Uhlenbeck semigroup*, J. Anal. Math. 78 (1999), 401-407.

➢ $T_m^H(f)(x) = e^{-x^2/2}\left(T_M^{\mathbf{O}}(e^{y^2/2}f)(x) + A_\phi(e^{y^2/2}f)(x)\right)$, a.e. $x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$.

➢ A_ϕ is bounded on $L^p(\mathbb{R}, e^{-x^2}dx)$, $p \in (1,\infty)$, and from $L^1(\mathbb{R}, e^{-x^2}dx)$ into $L^{1,\infty}(\mathbb{R}, e^{-x^2}dx)$.

Main steps in the proof of Theorem 1: Laguerre multiplier

Proposition ("kernels")

Let $\phi \in L^\infty(0,\infty)$. Then, there exists $C > 0$ such that:

(a) $|K_\phi^{\mathbf{H}}(x,y)| \le C \frac{||\phi||_{L^\infty(0,\infty)}}{\max\{x,y\}}$, $0 < y < \frac{x}{2}$ or $0 < 2x < y$.

(b) $|K_\phi^{\mathbf{H}}(x,y)| \le C \frac{||\phi||_{L^\infty(0,\infty)}}{|x|+|y|}$, $xy < 0$.

(c) $|K_\phi^{\mathbf{L}\alpha}(x,y)| \le C||\phi||_{L^\infty(0,\infty)} \frac{(\min\{x,y\})^{\alpha+\frac{1}{2}}}{(\max\{x,y\})^{\alpha+\frac{3}{2}}}$, $0 < y < \frac{x}{2}$ or $0 < 2x < y$.

(d) $|K_\phi^{\mathbf{L}\alpha}(x,y) - K_\phi^{H}(x,y)| \le C \frac{||\phi||_{L^\infty(0,\infty)}}{x} \left(1 + \sqrt{\frac{x}{|y-x|}}\right)$, $0 < \frac{x}{2} < y < 2x$.

- $\langle T_m^{\mathbf{L}\alpha}(f), g\rangle_{L^2(0,\infty)} = \langle \int_0^\infty \phi(t) \frac{\partial}{\partial t}[W_t^{\mathbf{L}\alpha}(f)(x) - W_{t,+}^{\mathbf{H}}(f)(x)]dt, g(x)\rangle_{L^2(0,\infty)}$

$$+\langle T_m^{\mathbf{H}}(\tilde{f}), \tilde{g}\rangle_{L^2(\mathbb{R})}, \quad f,g \in C_c^\infty(0,\infty).$$

  - $\tilde{f}(x) = f(x)$, $x > 0$, $\tilde{f}(x) = 0$, $x \le 0$ and $W_{t,+}^{\mathbf{H}}(f) = W_t^{\mathbf{H}}(\tilde{f})$.

- Theorem 1 for $T_m^{\mathbf{H}}$.

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- Theorem 1 for $T_m^{\mathbf{H}}$.

Main steps in the proof of Theorem 1: Schrödinger setting

Let $f, g \in C_c^\infty(\mathbb{R}^n)$.

- $\langle T_m^{\mathscr{L}}(f), g\rangle_{L^2(\mathbb{R}^n)} = \int_{[0,\infty)} m(\lambda) d\mu_{f,g;\mathscr{L}}(\lambda)$.
  - $\mu_{f,g;\mathscr{L}}$ is the complex measure given by

$$\mu_{f,g;\mathscr{L}}(A) = \langle E_{\mathscr{L}}(A)f, g\rangle, \quad A \text{ Borel set in } [0,\infty).$$

- Spectral multiplier associated to $(-\Delta)$: $T_m^{(-\Delta)}(f) = (m(|y|^2)\hat{f})^{\vee}$, $f \in L^2(\mathbb{R}^n)$.
  - $T_m^{(-\Delta)}(f)(x) = \lim_{\varepsilon \to 0^+} \left(\alpha(\varepsilon) f(x) + \int_{|x-y|>\varepsilon} K_\phi^{(-\Delta)}(x,y) f(y) dy \right)$, a.e. $x \in \mathbb{R}^n$.
    - $\alpha(\varepsilon) = \frac{1}{\Gamma(n/2)} \int_0^\infty \phi(\frac{\varepsilon^2}{4u}) e^{-u} u^{n/2-1} du, \quad \varepsilon > 0.$
    - If there exists $\phi(0^+)$, then $\lim_{\varepsilon\to 0^+} \alpha(\varepsilon) = \phi(0^+)$.
- $\langle T_m^{\mathscr{L}}(f), g\rangle_{L^2(\mathbb{R}^n)} = \langle \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} (K_\phi^{\mathscr{L}}(x,y) - K_\phi^{(-\Delta)}(x,y)) f(y) dy, g\rangle_{L^2(\mathbb{R}^n)}$

$$+ \langle T_m^{(-\Delta)}(f), g\rangle_{L^2(\mathbb{R}^n)}.$$

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  - ➛ $\alpha(\varepsilon) = \frac{1}{\Gamma(n/2)}\int_0^\infty \phi(\frac{\varepsilon^2}{4u}) e^{-u} u^{n/2-1} du, \quad \varepsilon > 0.$
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Extension to L^p spaces as Principal Value Operators

Denote $(\mathbb{L},\Omega)$=$(\mathbf{H},\mathbb{R}),(\mathbf{L}_\alpha,(0,\infty))$ or $(\mathscr{L},\mathbb{R}^n)$, $n\geq 3$.

Theorem 2

Let m a function of Laplace transform type associated to $\phi\in L^\infty(0,\infty)$. Then, $T_m^{\mathbb{L}}$ can be extended to $L^p(\Omega)$, $1\leq p<\infty$, as a bounded operator from $L^p(\Omega)$ into itself, when $p\in(1,\infty)$, and from $L^1(\Omega)$ into $L^{1,\infty}(\Omega)$.
This extension can be given by

$$T_m^{\mathbb{L}}(f)(x)=\lim_{\varepsilon\to 0^+}\left(\Lambda(\varepsilon)f(x)+\int_{|x-y|>\varepsilon}K_\phi^{\mathbb{L}}(x,y)f(y)dy\right),\ \text{a.e. } x\in\Omega,$$

If there exists the limit: $\lim_{t\to 0^+}\phi(t)=\phi(0^+)$, then

$$T_m^{\mathbb{L}}(f)(x)=\phi(0^+)f(x)+\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}K_\phi^{\mathbb{L}}(x,y)f(y)dy,\ \text{a.e. } x\in\Omega,$$

for every $f\in L^p(\Omega)$, $1\leq p<\infty$.

Imaginary powers on Banach-valued spaces

Denote $(\mathbb{L},\Omega)$= $(\mathbf{H},\mathbb{R})$, $(\mathbf{L}_\alpha,(0,\infty))$ or $(\mathscr{L},\mathbb{R}^n)$, $n\geq 3$.

➢ Consider $\gamma\in\mathbb{R}$. The imaginary power $\mathbb{L}^{i\gamma}$ is the multiplier of Laplace transform given by $\mathbb{L}^{i\gamma}=T^{\mathbb{L}}_{m_\gamma}$ where

$$m_\gamma(\lambda)=\lambda^{i\gamma}=\lambda\int_0^\infty e^{-\lambda t}\phi_\gamma(t)dt,\quad \lambda>0, \text{ and}$$

$$\phi_\gamma(t)=\frac{t^{-i\gamma}}{\Gamma(1-i\gamma)}\in L^\infty(0,\infty).$$

➢ By Theorem 2, $\mathbb{L}^{i\gamma}$ is bounded from $L^p(\Omega)$ into itself, for every $p\in(1,\infty)$.

Let $\mathbb{B}$ a Banach space. We can define $\mathbb{L}^{i\gamma}$ on $L^p(\Omega)\otimes\mathbb{B}$ in the natural way.

Can we extend $\mathbb{L}^{i\gamma}$ boundedly to the Bochner- Lebesgue space $L^p_{\mathbb{B}}(\Omega)$?

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Characterization of UMD spaces

$(\mathbb{L},\Omega)$= $(\mathbf{H},\mathbb{R})$, $(\mathbf{L}_\alpha,(0,\infty))$ or $(\mathscr{L},\mathbb{R}^n)$, $n \geq 3$.

Theorem 3

Let $\mathbb{B}$ a Banach space. The following assertions are equivalent.
(a) $\mathbb{B}$ is UMD.
(b) For every $\gamma \in \mathbb{R}$, and for some (equiv., for any) $p \in (1,\infty)$, $\mathbb{L}^{i\gamma}$ can be extended as a bounded operator from $L^p_{\mathbb{B}}(\Omega)$ into itself.

Main steps in the proof of Theorem 3: Hermite setting

$$M_\gamma(\lambda) := \lambda \int_0^\infty e^{-(\lambda+1/2)t} \phi_\gamma(t)\,dt, \quad \lambda > 0.$$

- $\mathbb{B}$ is UMD $\iff$ $(-\Delta)^{i\gamma}$ extends boundedly to $L^p_{\mathbb{B}}(\mathbb{R})$, $p \in (1,\infty)$.
  (Guerre-Delabriére and McConnell's result)
  $\iff$ $T^{\mathbf{O}}_{M_\gamma}$ extends boundedly to $L^p_{\mathbb{B}}(\mathbb{R}, e^{-x^2}dx)$.
  (comparison in local region)
- $T^{\mathbf{O}}_{M_\gamma}$ extends boundedly to $L^2_{\mathbb{B}}(\mathbb{R}, e^{-x^2}dx)$.
  $\iff$ $T^{\mathbf{H}}_{m_\gamma}$ extends boundedly to $L^2_{\mathbb{B}}(\mathbb{R})$.
  (relation Hermite and Ornstein-Uhlenbeck context)
  $\iff$ $T^{\mathbf{H}}_{m_\gamma}$ extends boundedly to $L^p_{\mathbb{B}}(\mathbb{R})$, $p \in (1,\infty)$.
  (It is a vector-valued C-Z operator)

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Main steps in the proof of Theorem 3: Laguerre setting

Let $f \in C_c^\infty(0,\infty) \otimes \mathbb{B}$,and denote by $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{B}$ the function such that:

$$\tilde{f}(x) = f(x), x > 0 \quad \text{and} \quad \tilde{f}(x) = 0,\ x \leq 0.$$

We decompose $T_{m_\gamma}^{\mathbf{L}_\alpha}$ as follows:

$$T_{m_\gamma}^{\mathbf{L}_\alpha}(f) = T_{m_\gamma,\mathrm{glob}}^{\mathbf{L}_\alpha}(f) + D_{m_\gamma}^{\alpha}(f) + T_{m_\gamma}^{\mathbf{H}}(\tilde{f}) - T_{m_\gamma,\mathrm{glob}}^{\mathbf{H}}(f),$$

where

- $T_{m_\gamma,\mathrm{glob}}^{\mathbf{L}_\alpha}(f)(x) = T_{m_\gamma}^{\mathbf{L}_\alpha}(f\chi_{(0,\frac{x}{2})\cup(2x,\infty)})(x)$, $x \in (0,\infty)$,
- $D_{m_\gamma}^{\alpha}(f)(x) = \int_{x/2}^{2x} [K_{\phi_\gamma}^{L_\alpha}(x,y) - K_{\phi_\gamma}^{H}(x,y)] f(y)dy$, $x \in (0,\infty)$,
- $T_{m_\gamma,\mathrm{glob}}^{\mathbf{H}}(f)(x) = T_{m_\gamma}^{\mathbf{H}}(f\chi_{(0,\frac{x}{2})\cup(2x,\infty)})(x)$, $x \in (0,\infty)$.

➢ By using Proposition "kernels" we obtain that:

$T_{m_\gamma,\mathrm{glob}}^{\mathbf{L}_\alpha}$, $D_{m_\gamma}^{\alpha}$ and $T_{m_\gamma,\mathrm{glob}}^{\mathbf{H}}$ can be extended boundedly on $L_{\mathbb{B}}^p(0,\infty)$.

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- $D_{m_\gamma}^{\alpha}(f)(x) = \int_{x/2}^{2x}[K_{\phi_\gamma}^{L_\alpha}(x,y) - K_{\phi_\gamma}^{H}(x,y)]f(y)dy$, $x \in (0,\infty)$,
- $T_{m_\gamma,\mathrm{glob}}^{\mathbf{H}}(f)(x) = T_{m_\gamma}^{\mathbf{H}}(f\chi_{(0,\frac{x}{2})\cup(2x,\infty)})(x)$, $x \in (0,\infty)$.

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Main steps in the proof of Theorem 3: Laguerre setting

$$T^{\mathbf{L}_\alpha}_{m_\gamma}(f) = T^{\mathbf{L}_\alpha}_{m_\gamma,\mathrm{glob}}(f) + D^{\alpha}_{m_\gamma}(f) + T^{\mathbf{H}}_{m_\gamma}(\tilde{f}) - T^{\mathbf{H}}_{m_\gamma,\mathrm{glob}}(f), \quad f \in C_c^\infty(0,\infty) \otimes \mathbb{B}.$$

$(a)_{UMD} \Rightarrow (b)_{L^p}$ It follows by using Theorem 3 for the Hermite operator.

$(b)_{L^p} \Rightarrow (a)_{UMD}$ $\quad ||T^{\mathbf{H}}_{m_\gamma}(\tilde{f})||_{L^p_B(\mathbb{R})} \leq C||f||_{L^p_{\mathbb{B}}(0,\infty)}, \; f \in L^p(0,\infty) \otimes \mathbb{B}.$

Let $f \in L^p(\mathbb{R}) \otimes B$. Decompose f as $f = f_1 + f_2$, where $f_1 = f\chi_{(0,\infty)}$.

$$\begin{aligned} ||T^{\mathbf{H}}_{m_\gamma}(f)||_{L^p_B(\mathbb{R})} &\leq \sum_{j=1}^{2} \Big(||\chi_{(0,\infty)} T_{m_\gamma}(f_j)||_{L^p_B(\mathbb{R})} + ||\chi_{(-\infty,0]} T_{m_\gamma}(f_j)||_{L^p_B(\mathbb{R})} \Big) \\ &\leq C||f||_{L^p_B(\mathbb{R})}. \end{aligned}$$

Main steps in the proof of Theorem 3: Laguerre setting

$$T_{m_\gamma}^{\mathbf{L}_\alpha}(f) = T_{m_\gamma,\mathrm{glob}}^{\mathbf{L}_\alpha}(f) + D_{m_\gamma}^{\alpha}(f) + T_{m_\gamma}^{\mathbf{H}}(\tilde{f}) - T_{m_\gamma,\mathrm{glob}}^{\mathbf{H}}(f), \quad f \in C_c^\infty(0,\infty) \otimes \mathbb{B}.$$

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Main steps in the proof of Theorem 3: Schrödinger setting

- We split $T_{m_\gamma}^{(-\Delta)} = T_{m_\gamma,\mathrm{loc}}^{(-\Delta)} + T_{m_\gamma,\mathrm{glob}}^{(-\Delta)}$, where

$$T_{m_\gamma,\mathrm{glob}}^{(-\Delta)}(f)(x) = \int_{|x-y|\geq\rho(x)} K_{\phi_\gamma}^{(-\Delta)}(x,y)f(y)dy,\ x \in \mathbb{R}^n.$$

$$\rho(x) = \sup\{r > 0 : r^{2-n} \int_{B(x,r)} V(y)dy \leq 1\},\ x \in \mathbb{R}^n.$$

- For every $f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$, we write

$$T_{m_\gamma}^{\mathscr{L}}(f) = \overbrace{T_{m_\gamma}^{\mathscr{L}}(f) - T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}}^{S_{m_\gamma}(f)} + T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}(f).$$

- S_{m_γ} can be extended as a bounded operator on $L^p_{\mathbb{B}}(\mathbb{R}^n)$, $p \in (1,\infty)$.

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$$T_{m_\gamma}^{\mathcal{L}}(f) = S_{m_\gamma}(f) + T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}(f), \quad f \in C_c^\infty(\mathbb{R}^n) \otimes B.$$

(a) $_{UMD}$ $\Rightarrow$(b) $_{L^p}$ By using that $\mathbb{B}$ is UMD and $T_{m_\gamma}^{(-\Delta)}$ is a C-Z operator:

$T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}(f)$ can be extended boundedly to $L_{\mathbb{B}}^p(\mathbb{R}^n)$, $p \in (1,\infty)$.

(b) $_{L^p}$ $\Rightarrow$ (a) $_{UMD}$ Inspired in arguments developed in:

✎ I. Abu-Falahah, P.R. Stinga and J.L. Torrea, *Square functions associated to Schödinger operators*, Studia Math. 203 (2011), 171-194.

we establish that:

$L_{\mathbb{B}}^p$-boundedness for $T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}$ implies $L_{\mathbb{B}}^p$-boundedness for $T_{m_\gamma}^{(-\Delta)}$.

Main steps in the proof of Theorem 3: Schrödinger setting

$$T_{m_\gamma}^{\mathscr{L}}(f) = S_{m_\gamma}(f) + T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}(f), \quad f \in C_c^\infty(\mathbb{R}^n) \otimes B.$$

(a) $_{UMD}$ $\Rightarrow$(b) $_{L^p}$ By using that $\mathbb{B}$ is UMD and $T_{m_\gamma}^{(-\Delta)}$ is a C-Z operator:

$$T_{m_\gamma,\mathrm{loc}}^{(-\Delta)}(f) \text{ can be extended boundedly to } L^p_{\mathbb{B}}(\mathbb{R}^n),\ p \in (1,\infty).$$

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Thank you very much!