Variable exponent Campanato spaces

Humberto Rafeiro

Instituto Superior Técnico, Lisbon

FSDONA2011, *Tabarz/Thür*

Outline

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$
   - modified Diening's inequality

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$
   - modified Diening's inequality
4. Variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$
   - modified Diening's inequality
4. Variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$
5. Variable exponent Campanato spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$

Outline

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$
   - modified Diening's inequality
4. Variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$
5. Variable exponent Campanato spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$
   - Embedding
   - Coincidence with $L^{p(\cdot),\lambda(\cdot)}(X)$

Outline

1. Spaces of homogeneous type
2. Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$
3. Variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$
   - modified Diening's inequality
4. Variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$
5. Variable exponent Campanato spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$
   - Embedding
   - Coincidence with $L^{p(\cdot),\lambda(\cdot)}(X)$
   - Coincidence with $H^{\alpha(\cdot)}(X)$

quasimetric

function $d : X \times X \to [0, \infty)$ which satisfies the usual metric axioms with the triange inequality replaced by the *quasi-triangle inequality*

$$d(x,y) \leq \mathcal{Q}[d(x,z) + d(z,y)], \quad \mathcal{Q} \geq 1 \tag{1}$$

Ahlfors regularity

lower Ahlfors α-regular, if

$$\mu B(x, r) \geq c r^{\alpha} \tag{2}$$

Ahlfors regularity

lower Ahlfors α-regular, if

$$\mu B(x, r) \geq c r^{\alpha} \tag{2}$$

upper Ahlfors β-regular, if

$$\mu B(x, r) \leq c r^{\beta}, \tag{3}$$

where $\alpha, \beta, c > 0$ does not depend on x and r.

doubling condition

$$\mu B(x, 2r) \leq \mathcal{D} \mu B(x, r), \quad \mathcal{D} > 1 \tag{4}$$

with $\mathcal{D}$ not depending on $x \in X$ and $0 < r < d_X$

doubling condition

$$\mu B(x, 2r) \leq \mathcal{D} \mu B(x, r), \quad \mathcal{D} > 1 \tag{4}$$

with $\mathcal{D}$ not depending on $x \in X$ and $0 < r < d_X$

$$\frac{\mu B(x, R)}{\mu B(y, r)} \leq \mathcal{D} \left(\frac{R}{r} \right)^N, \quad 0 < r \leq R \tag{5}$$

for all d-balls $B(x, R)$ and $B(y, r)$ with $B(y, r) \subset B(x, R)$, where $N = \log_2 \mathcal{D}$ is called the *doubling order of* μ.

spaces of homogeneous type

spaces of homogeneous type

- μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x,r)$. Everywhere in the sequel we suppose that all the balls have a finite measure, that is, $\mu B(x,r) < \infty$ for all $x \in X$ and $r > 0$. and that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$.

spaces of homogeneous type

- μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r)$. Everywhere in the sequel we suppose that all the balls have a finite measure, that is, $\mu B(x, r) < \infty$ for all $x \in X$ and $r > 0$. and that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$.
- triplet (X, d, μ), with μ satisfying the doubling condition, is said a *space of homogeneous type*

Reverse doubling condition

Let (X, d, μ) be a space of homogeneous type. If

$$\mu\left(B(x,R)\backslash B(x,r)\right) > 0 \tag{6}$$

Reverse doubling condition

Let (X, d, μ) be a space of homogeneous type. If

$$\mu\left(B(x,R)\backslash B(x,r)\right) > 0 \tag{6}$$

is valid, then the measure μ satisfies the reverse doubling condition

$$\frac{\mu B(x,r)}{\mu B(x,R)} \leq C\left(\frac{r}{R}\right)^{\gamma} \tag{7}$$

for all $x \in X$ and $0 < r \leq R < d_X$, where $C, \gamma > 0$.

Definition of variable exponent Lebesgue spaces

$$1 \leq p_- \leq p(x) \leq p_+ < \infty$$

Definition of variable exponent Lebesgue spaces

$$1 \leq p_- \leq p(x) \leq p_+ < \infty$$

$$I^{p(\cdot)}(f) := \int_X |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \tag{8}$$

Definition of variable exponent Lebesgue spaces

$$1 \le p_- \le p(x) \le p_+ < \infty$$

$$I^{p(\cdot)}(f) := \int_X |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \tag{8}$$

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I^{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}, \tag{9}$$

Definition of variable exponent Lebesgue spaces

$$1 \leq p_- \leq p(x) \leq p_+ < \infty$$

$$I^{p(\cdot)}(f) := \int_X |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \tag{8}$$

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I^{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}, \tag{9}$$

modular vs. norm

$$\|f\|_{p(\cdot)}^{\theta} \leq I^{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}^{\sigma} \tag{10}$$

Definition of variable exponent Lebesgue spaces

$$1 \leq p_- \leq p(x) \leq p_+ < \infty$$

$$I^{p(\cdot)}(f) := \int_X |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \tag{8}$$

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I^{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}, \tag{9}$$

modular vs. norm

$$\|f\|_{p(\cdot)}^{\theta} \leq I^{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}^{\sigma} \tag{10}$$

where

$$\theta = \begin{cases} p_+, \text{ if } \|f\|_{p(\cdot)} \leq 1 \\ p_-, \text{ if } \|f\|_{p(\cdot)} \geq 1 \end{cases} \quad \text{and} \quad \sigma = \begin{cases} p_-, \text{ if } \|f\|_{p(\cdot)} \leq 1 \\ p_+, \text{ if } \|f\|_{p(\cdot)} \geq 1 \end{cases} . \tag{11}$$

Hölder's inequality

$$\int_X |f(x)\varphi(x)| \,\mathrm{d}\mu(x) \leq \left(\frac{1}{p_-} + \frac{1}{p_+} \right) \|f\|_{p(\cdot)} \|\varphi\|_{p'(\cdot)}. \tag{12}$$

Hölder's inequality

$$\int_X |f(x)\varphi(x)|\,\mathrm{d}\mu(x) \le \left(\frac{1}{p_-}+\frac{1}{p_+}\right)\|f\|_{p(\cdot)}\|\varphi\|_{p'(\cdot)}. \tag{12}$$

Embedding

$\mu X < \infty$ and $1 \le p(x) \le q(x) \le q^+ < \infty$, we have the embedding

$$L^{q(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X). \tag{13}$$

usual log-Hölder condition $\mathcal{P}^{\log}(X)$

$$|p(x) - p(y)| \leq \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leq \frac{1}{2}, \quad x, y \in X, \qquad (14)$$

usual log-Hölder condition $\mathcal{P}^{\log}(X)$

$$|p(x) - p(y)| \leq \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leq \frac{1}{2}, \quad x,y \in X, \qquad (14)$$

where $C_p > 0$ does not depend on x and y

usual log-Hölder condition $\mathcal{P}^{\log}(X)$

$$|p(x) - p(y)| \leq \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leq \frac{1}{2}, \quad x, y \in X, \qquad (14)$$

where $C_p > 0$ does not depend on x and y

μ log-Hölder condition $\mathcal{P}^{\log}_{\mu}(X)$

usual log-Hölder condition $\mathcal{P}^{\log}(X)$

$$|p(x) - p(y)| \le \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \le \frac{1}{2}, \quad x, y \in X, \tag{14}$$

where $C_p > 0$ does not depend on x and y

μ log-Hölder condition $\mathcal{P}^{\log}_{\mu}(X)$

the class $\mathcal{P}^{\log}_{\mu}(X)$ of functions $p : X \to [1, \infty)$ satisfying the condition

$$|p(x) - p(y)| \le \frac{A}{-\ln \mu B(x, d(x,y))} \tag{15}$$

for all $x, y \in X$ such that $\mu B(x, d(x,y)) < \frac{1}{2}$.

Lemma

If (X, d, μ) *has the lower Ahlfors property, then* $\mathcal{P}^{\log}(X) \subseteq \mathcal{P}^{\log}_{\mu}(X)$.

Lemma

If (X, d, μ) *has the lower Ahlfors property, then* $\mathcal{P}^{\log}(X) \subseteq \mathcal{P}^{\log}_\mu(X)$.

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure and* $p \in \mathcal{P}^{\log}_\mu(X)$. *Then*

$$C^{-1}\mu B(x,r) \leq (\mu B(x,r))^{\frac{p(x)}{p(y)}} \leq C\mu B(x,r) \tag{16}$$

for all $x, y \in X$ *such that* $y \in B(x,r)$, *with the constant* $C \geq 1$ *not depending on* x, y, r.

Lemma

If (X, d, μ) *has the lower Ahlfors property, then* $\mathcal{P}^{\log}(X) \subseteq \mathcal{P}^{\log}_{\mu}(X)$.

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure and* $p \in \mathcal{P}^{\log}_{\mu}(X)$. *Then*

$$C^{-1}\mu B(x,r) \le (\mu B(x,r))^{\frac{p(x)}{p(y)}} \le C\mu B(x,r) \tag{16}$$

for all $x, y \in X$ *such that* $y \in B(x,r)$, *with the constant* $C \ge 1$ *not depending on* x, y, r.

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure and* $p \in \mathcal{P}^{log}_{\mu}(X)$. *Then*

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \sim (\mu B(x,r))^{\frac{1}{p(x)}}. \tag{17}$$

Definition of variable exponent Morrey space

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \quad (18)$$

Definition of variable exponent Morrey space

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} |f(y)|^{p(y)} \,\mathrm{d}\mu(y) < \infty \quad (18)$$

Norms

$$\|f\|_1 = \inf\left\{\lambda > 0 : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\} \quad (19)$$

Definition of variable exponent Morrey space

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} |f(y)|^{p(y)} \, \mathrm{d}\mu(y) < \infty \quad (18)$$

Norms

$$\|f\|_1 = \inf\left\{\lambda > 0 : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\} \quad (19)$$

$$\|f\|_2 = \sup_{x\in X, r>0} \left\| (\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{B(x,r)} \right\|_{p(\cdot)} \quad (20)$$

Definition of variable exponent Morrey space

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} |f(y)|^{p(y)} \,\mathrm{d}\mu(y) < \infty \quad (18)$$

Norms

$$\|f\|_1 = \inf\left\{\lambda > 0 : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\} \quad (19)$$

$$\|f\|_2 = \sup_{x\in X, r>0} \left\| (\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{B(x,r)} \right\|_{p(\cdot)} \quad (20)$$

If $p \in \mathcal{P}_\mu^{\log}(X)$

$$\|f\|^*_{L^{p(\cdot),\lambda(\cdot)}(X)} = \sup_{x\in X, r>0} (\mu B(x,r))^{-\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r))}. \quad (21)$$

Lemma

Let (X, d, μ) *be Q.M.M.S. with finite measure,* $p, q \in \mathcal{P}_\mu^{\log}(X)$*,* $0 \le \lambda(x) \le 1$ *and* $0 \le \nu(x) \le 1$*. Let also* $1 \le p(x) \le q(x)$ *and*

$$\frac{1-\lambda(x)}{p(x)} \ge \frac{1-\nu(x)}{q(x)}. \tag{22}$$

Then

$$L^{q(\cdot),\nu(\cdot)}(X) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(X). \tag{23}$$

Diening's inequality in variable exponent Lebesgue spaces

For $f \in L^{p(\cdot)}(\Omega)$, $\int_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}y \leq 1$ and $p \in \mathcal{P}^{\log}(\Omega)$

$$\left(\fint_{B(x,r)} |f(y)|\,\mathrm{d}y\right)^{p(x)} \leq C\left(1 + \fint_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}y\right) \tag{24}$$

Diening's inequality in variable exponent Lebesgue spaces

For $f \in L^{p(\cdot)}(\Omega)$, $\int_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}y \leq 1$ and $p \in \mathcal{P}^{\log}(\Omega)$

$$\left(\fint_{B(x,r)} |f(y)|\,\mathrm{d}y \right)^{p(x)} \leq C \left(1 + \fint_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}y \right) \tag{24}$$

Lemma

Let (X, d, μ) be a Q.M.M.S. with finite measure satisfying the annular-measure property, let $0 \leq \lambda(x) \leq 1$ and $p \in \mathcal{P}^{\log}_{\mu}(X)$. Then

$$\left(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y) \right)^{p(z)} \leq C \left(1 + \fint_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}\mu(y) \right) \tag{25}$$

for all $z \in B(x, r)$, provided $\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \leq 1$.

Definition of variable exponent Hölder spaces

Let $\alpha(x)$ be a μ-measurable real-valued non-negative function on X. We say that a bounded function f belongs to $H^{\alpha(\cdot)}(X)$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C \cdot d(x, y)^{\max\{\alpha(x),\alpha(y)\}} \tag{26}$$

for every $x, y \in X$.

Definition of variable exponent Hölder spaces

Let $\alpha(x)$ be a μ-measurable real-valued non-negative function on X. We say that a bounded function f belongs to $H^{\alpha(\cdot)}(X)$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C \cdot d(x, y)^{\max\{\alpha(x), \alpha(y)\}} \tag{26}$$

for every $x, y \in X$. This is a Banach space with respect to the norm

$$\|f\|_{H^{\alpha(\cdot)}(X)} = \|f\|_{L^\infty} + \lfloor f \rfloor_{\alpha(\cdot)}, \tag{27}$$

where

$$\lfloor f \rfloor_{\alpha(\cdot)} := \sup_{x,y \in X} \frac{|f(x) - f(y)|}{d(x, y)^{\max\{\alpha(x), \alpha(y)\}}}. \tag{28}$$

Definition of variable exponent Campanato spaces

$$\mathcal{J}^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^{p(y)} \mathrm{d}\mu(y) < \infty, \tag{29}$$

Definition of variable exponent Campanato spaces

$$\mathcal{J}^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^{p(y)} \mathrm{d}\mu(y) < \infty, \tag{29}$$

$$_1\lfloor f \rfloor^{p(\cdot)}_{\lambda(\cdot)} := \inf\left\{\eta > 0 : \mathcal{J}^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\}, \tag{30}$$

Definition of variable exponent Campanato spaces

$$\mathcal{J}^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^{p(y)} \mathrm{d}\mu(y) < \infty, \tag{29}$$

$$_1\lfloor f \rfloor_{\lambda(\cdot)}^{p(\cdot)} := \inf\left\{\eta > 0 : \mathcal{J}^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\}, \tag{30}$$

$$_2\lfloor f \rfloor_{\lambda(\cdot)}^{p(\cdot)} := \sup_{x\in X; r>0} \left\| (\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} \left(f - f_{B(x,r)}\right) \chi_{B(x,r)} \right\|_{L^{p(\cdot)}(X)}. \tag{31}$$

Lemma

Let (X, d, μ) *be a Q.M.M.S.. For every function* $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$*, the inequalities*

$$\lfloor f \rfloor_i^{\theta} \leq \mathcal{I}^{p(\cdot),\lambda(\cdot)}(f) \leq \lfloor f \rfloor_i^{\sigma}, \quad i = 1, 2; \tag{32}$$

are valid, where

$$\theta = \begin{cases} p_+, \text{ if } \lfloor f \rfloor_i \leq 1 \\ p_-, \text{ if } \lfloor f \rfloor_i \geq 1 \end{cases} \quad \text{and} \quad \sigma = \begin{cases} p_-, \text{ if } \lfloor f \rfloor_i \leq 1 \\ p_+, \text{ if } \lfloor f \rfloor_i \geq 1 \end{cases}.$$

Lemma

Let (X, d, μ) *be a Q.M.M.S.. For* $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$ *we have* $\lfloor f \rfloor_1 = \lfloor f \rfloor_2$.

Lemma

Let (X, d, μ) be a Q.M.M.S.. For $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$ we have $\lfloor f \rfloor_1 = \lfloor f \rfloor_2$.

Definition

The variable exponent Campanato space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$ will be endowed with the following norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)} := {}_1\lfloor f \rfloor^{p(\cdot)}_{\lambda(\cdot)} + \|f\|_{L^1(X)}. \tag{33}$$

Due to the fact that the semi-norms $\lfloor \cdot \rfloor_1$ and $\lfloor \cdot \rfloor_2$ coincide, we can take either $\lfloor f \rfloor_2$ or $\lfloor f \rfloor_1$ in (33).

Imbedding theorem

Let (X, d, μ) be Q.M.M.S. with finite measure and λ and ν be non-negative bounded functions. If $p, q \in \mathcal{P}^{\log}_{\mu}(X)$, $1 \leq p(x) \leq q(x) \leq q_+ < \infty$, and

$$\frac{1-\lambda(x)}{p(x)} \geq \frac{1-\nu(x)}{q(x)}, \tag{34}$$

then

$$\mathcal{L}^{q(\cdot),\nu(\cdot)}(X) \hookrightarrow \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X). \tag{35}$$

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure,* $p \in \mathcal{P}_{\mu}^{\log}(X)$ *and* $0 \le \lambda(x) \le 1$. *For* $f \in L^{p(\cdot),\lambda(\cdot)}(X)$ *with* $I^{p(\cdot),\lambda(\cdot)}(f) \le 1$, *the estimate*

$$\mathcal{J}^{p(\cdot),\lambda(\cdot)}(f) \le C \left[I^{p(\cdot),\lambda(\cdot)}(f) + \sup_{x \in X, r>0} (\mu B(x,r))^{1-\lambda(x)} \right] \tag{36}$$

holds, where C *does not depend on* f *and* C.

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure,* $p \in \mathcal{P}^{\log}_{\mu}(X)$ *and* $0 \le \lambda(x) \le 1$. *For* $f \in L^{p(\cdot),\lambda(\cdot)}(X)$ *with* $I^{p(\cdot),\lambda(\cdot)}(f) \le 1$, *the estimate*

$$\mathcal{J}^{p(\cdot),\lambda(\cdot)}(f) \le C \left[I^{p(\cdot),\lambda(\cdot)}(f) + \sup_{x \in X, r > 0} (\mu B(x,r))^{1-\lambda(x)} \right] \qquad (36)$$

holds, where C *does not depend on* f *and* C.

Corollary

Let (X, d, μ) *be a Q.M.M.S. with finite measure,* $p \in \mathcal{P}^{\log}_{\mu}(X)$ *and* $0 \le \lambda(x) \le 1$. *Then*

$$L^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X). \qquad (37)$$

Lemma

Let (X, d, μ) *be a Q.M.M.S. with finite measure. Then there exists a constant* C *such that*

$$|f_{B(x,\rho)} - f_{B(x,\sigma)}| \leq C \left(\frac{(\mu B(x,\rho))^{\lambda(x)} + (\mu B(x,\sigma))^{\lambda(x)}}{\mu B(x,\sigma)} \right)^{\frac{1}{p(x)}} \times \|f\|^{*}_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)}, \quad (38)$$

for all $x \in X$ *and* $0 < \sigma < \rho < d_X$.

Lemma

Let (X, d, μ) *be a S.H.T. with finite measure and* λ *be a non-negative real-valued function with* $\lambda_+ < 1$*. Then there exists a constant* $C = C(p, \lambda, \mathcal{D})$ *such that*

$$|f_{B(x,r)} - f_{B(x,r/2^m)}| \leq C \|f\|^*_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \sum_{k=0}^{m-1} \mathcal{D}^{kN\frac{1-\lambda(x)}{p(x)}} \quad (39)$$

for all $(x, r) \in X \times (0, d_X)$*, where* $\mathcal{D}$ *is the constant from the doubling condition and* $N = \log_2 \mathcal{D}$ *is the exponent from the iterated doubling condition.*

Lemma

Let (X, d, μ) *be a S.H.T. with finite measure, let the annular-measure property hold and* λ *be a non-negative real-valued function with* $\lambda_+ < 1$*. Then there exists a constant* $C = C(\mathcal{D}, p, \lambda) > 0$ *such that for any* $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$ *and all* $(x, \rho) \in X \times (0, d_X)$ *the inequality*

$$|f_{B(x,\rho)}| \leq |f_X| + C\|f\|^*_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)}(\mu B(x, \rho))^{\frac{\lambda(x)-1}{p(x)}} \qquad (40)$$

holds.

Lemma

Let (X, d, μ) be a S.H.T. with finite measure. For all $f \in \mathcal{L}^{1,\lambda(\cdot)}(X)$ and all $x, y \in \overline{X}$, there exists a constant $C = C(\mathcal{D}, \lambda)$, such that, for $r = 2\mathcal{Q}d(x, y)$ (where $\mathcal{Q}$ is the constant from (1)), we have

$$|f_{B(x,r)} - f_{B(y,r)}| \leq C\|f\|^{*}_{\mathcal{L}^{1,\lambda(\cdot)}(X)} \left[(\mu B(x, r))^{\lambda(x)-1} + (\mu B(y, r))^{\lambda(y)-1}\right]. \tag{41}$$

Theorem

Let (X, d, μ) *be a S.H.T. with finite measure, let* $p \in \mathcal{P}_{\mu}^{\log}(X)$ *and* λ *be a non-negative real-valued function with* $\lambda_+ < 1$*. Then*

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X) \cong L^{p(\cdot),\lambda(\cdot)}(X). \tag{42}$$

Lemma

Let (X, d, μ) be a Q.M.M.S. satisfying the reverse doubling condition and λ be a bounded real-valued function with $\lambda_- > 1$. Then there exists a constant $C = C(p, \lambda, \mathcal{D})$ such that

$$|f_{B(x,r)} - f_{B(x,r/2^m)}| \leq C\|f\|^*_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)}(\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \sum_{k=0}^{m-1} 2^{k\gamma\frac{1-\lambda(x)}{p(x)}} \quad (43)$$

for all $(x, r) \in X \times (0, d_X)$, where γ is the exponent from the reverse doubling condition.

$$S(\lambda) := \{x \in X : \lambda(x) = 1\}$$

$$X_\delta(\lambda) = \{x \in X : \lambda(x) \geq 1 + \delta\},\ \delta > 0.$$

$$S(\lambda) := \{x \in X : \lambda(x) = 1\}$$

$$X_\delta(\lambda) = \{x \in X : \lambda(x) \geq 1 + \delta\}, \ \delta > 0.$$

Lemma

Let (X, d, μ) be a Q.M.M.S. satisfying the reverse doubling condition and λ be a real-valued bounded function with values in $[1, \infty)$. If $\mu S(\lambda) = 0$, then for every $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)$ there exists a function $\widetilde{f}$ defined on $\overline{X}$ such that f equals $\widetilde{f}$ almost everywhere on X and

$$\lim_{r \to 0+} f_{B(x,r)} = \widetilde{f}(x) \tag{44}$$

for all $x \in \overline{X} \backslash S(\lambda)$, the convergence being uniform on every bounded subset of $X_\delta(\lambda)$ for every fixed $\delta > 0$.

Theorem

Let (X, d, μ) be a S.H.T. with finite measure, $p \in \mathcal{P}^{\log}(X)$ and λ be a bounded real-valued function with $\lambda_- > 1$. Then

$$H^{\alpha(\cdot)}(X) \hookrightarrow \mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)\,, \tag{45}$$

when α satisfy log-*Hölder continuity and*

$$\alpha(x) \geq N\frac{\lambda(x)-1}{p(x)}. \tag{46}$$

Theorem

Let (X, d, μ) *be a S.H.T. with finite measure, let the annular-measure property hold,* $p \in \mathcal{P}^{\log}(X)$ *and* λ *be a bounded real-valued function with* $\lambda_- > 1$*. Then*

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow H^{\alpha(\cdot)}(X) \tag{47}$$

when α *satisfy* log-*Hölder continuity condition and*

$$\alpha(x) \leq \gamma \frac{\lambda(x) - 1}{p(x)}. \tag{48}$$

Theorem

Let (X, d, μ) *be a S.H.T. with finite measure,* $p \in \mathcal{P}^{\log}(X)$, λ *is a real-valued function with* $\lambda_- > 1$, μ *is Ahlfors Q-regular and* λ *satisfy* log-*Hölder continuity condition . Then*

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X) \cong H^{\alpha(\cdot)}(X) \tag{49}$$

with

$$\alpha(x) = Q\frac{\lambda(x)-1}{p(x)}. \tag{50}$$

References

Almeida, A.; Hasanov, J.; Samko, S. (2008)
Maximal and potential operators in variable exponent Morrey spaces.
Georgian Math. J. 15(2), 195–208.

Fan, X. (2010)
Variable exponent Morrey and Campanato spaces.
Nonlinear Anal. 72, 4148–4161.

Kokilashvili, V.; Meskhi, A.(2008)
Boundedness of maximal and singular operators in Morrey spaces with variable exponent.
Armenian J. Math. 1, 18–28.

Genebashvili, I.; Gogatishvili, A.;Kokilashvili, V.; Krbec, M. (1998)
Weight theory for integral transforms on spaces of homogeneous type.
Pitman Monographs and Surveys in Pure and Applied Mathematics, 92. Longman.

Kufner, A.; John, O; Fučik, S. (1977)
Function spaces.
Noordhoff International Publishing, Leyden; Academia, Prague

The End