Quarkonial frames of wavelet type - Stability and moment conditions

Stephan Dahlke[1] Peter Oswald[2] Thorsten Raasch[3]

[1] Philipps-Universität Marburg
[2] Jacobs University Bremen
[3] Johannes Gutenberg-Universität Mainz

FSDONA 2011

Tabarz, September 18-24, 2011

Outline

Agenda

- Construction of frames
  - local support
    $\Rightarrow$ dilation, translation
  - high approximation order
    $\Rightarrow$ quarkonial frames
  - stability in $H^s(\mathbb{R})$, $s > 0$, $s = 0$ and $s < 0$
    $\Rightarrow$ moment conditions
- Applications
  - numerical solution of operator equations
    (differential or integral equations)
  - recovery algorithms for inverse problems
  - signal/image processing

Quarkonial frames

Triebel (1997ff.): quarkonial/subatomic decompositions

- combines Weierstraß approach for holomorphic functions

$$f(x) = \sum_{m=1}^{\infty} \sum_{p \in \mathbb{N}_0^n} c_{p,m} (x - x_m)^p \eta_m(x), \quad \sum_{m=1}^{\infty} \eta_m(x) = 1, \quad x \in \Omega$$

  with wavelet philosophy (dilation and translation)
- specifies conditions under which we have expansions

$$f(x) = \sum_{p \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} c_{p,j,k} \varphi_p(2^j x - k), \quad \varphi_p(x) = x^p \varphi(x), \quad x \in \Omega$$

  with

$$\|f\| \eqsim \inf \left\{ \|\mathbf{c}\| : \mathbf{c} \text{ expansion coefficients} \right\}$$

- other enrichment functions possible, e.g., orthogonal polynomials

Relations with other approaches

Crucial condition: partition of unity

$$\sum_{k \in \mathbb{Z}^n} \varphi(x-k) = 1$$

- close relation with partition of unity methods (PUM)

$$\sum_{\theta \in \Theta_m} \varphi_\theta(x) = 1, \quad x \in \Omega = \bigcup_{\theta \in \Theta_m} A_\theta(B)$$

$$\varphi_{p,\theta}(x) := (A_\theta(x))^p \varphi_\theta(x)$$

  see Babuška et al. (2003), Dahmen/Dekel/Petrushev (2007)
- other related topics:
  - stable space splittings (see previous talk of P. Oswald)
  - fusion frames, Casazza/Kutyniok et al. (2004ff.)

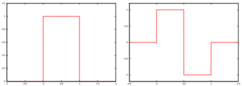
Special case: Discontinuous bases for $L_2(\mathbb{R})$

Let $\varphi(x) := \chi_{[0,1)}(x)$. Recall two classical ONBs for $L_2(\mathbb{R})$:

- **Haar wavelet basis** (Haar 1909)

$$\psi(x) := \varphi(2x) - \varphi(2x-1),$$

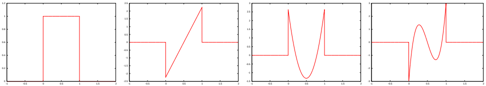
$$\Psi_H := \{\varphi(\cdot - k) : k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot - k) : j \geq 0, k \in \mathbb{Z}\}$$



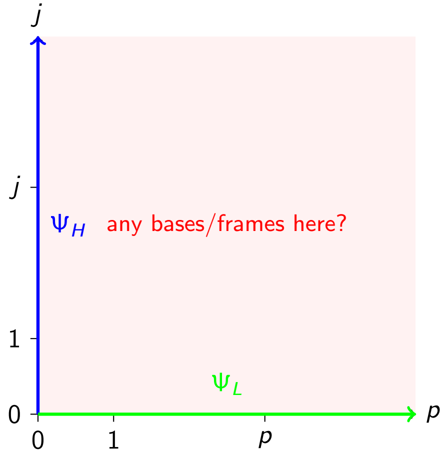
- **Legendre wavelet basis** (Alpert 1993, Razzaghi/Yousefi 2000)

$$\varphi_p(x) := \sqrt{2}L_p(2x-1)\varphi(x), \quad L_p \text{ } p\text{-th Legendre polynomial}$$

$$\Psi_L := \{\varphi_p(\cdot - k) : p \geq 0, k \in \mathbb{Z}\}$$



j-p diagram



Alpert multiwavelets

Consider $\varphi_p(x) := \sqrt{2} L_p(2x-1)\varphi(x)$, $p \geq 0$.

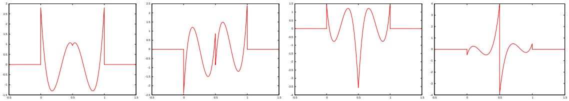
Observation

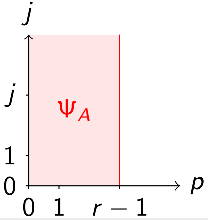
For fixed $j \geq 0$, $f \mapsto 2^{j/2} f(2^j \cdot)$ is an isometry. Therefore, $\{\varphi_{p,j,k} := 2^{j/2}\varphi_p(2^j \cdot -k) : p \geq 0, k \in \mathbb{Z}\}$ is an ONB.

Alpert (1993):

- orthonormal multiwavelet basis of length $r \in \mathbb{N}$
- $\Phi := (\varphi_p)_{0 \leq p < r}$, $\quad \Phi(x) = \sum_{k \in \mathbb{Z}} A_k \Phi(2x-k)$
- $\Psi = (\psi_p)_{0 \leq p < r} = \sum_{k \in \mathbb{Z}} B_k \Phi(2x-k)$ with orthonormal components

components ψ_p for $r = 4$:





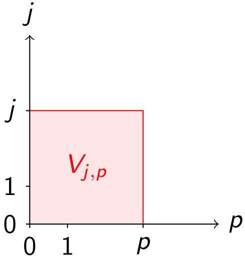
Stability of Legendre quarks in $H^s(\mathbb{R})$, $s > 0$

Goal: Prove the frame property of the Legendre quark system

$$\{\varphi_{p,j,k}(x) := 2^{j/2}\varphi_p(2^j\cdot - k), p, j \geq 0, k \in \mathbb{Z}\}, \quad \varphi_p(x) = \sqrt{2}L_p(2x-1)\varphi(x)$$

in $H^s(\mathbb{R})$, $s > 0$, when properly rescaled.

- we may use approximation and regularity properties of $V_{j,p} := \operatorname{clos}_{L_2(\mathbb{R})} \operatorname{span}\{\varphi_{i,j,k} : 0 \leq i \leq p, k \in \mathbb{Z}\}$



A generic frame theorem

Theorem

Let $\{\varphi_{i,j,k} : 0 \leq i < p, k \in \mathbb{Z}\} \subset H^\gamma(\mathbb{R})$ *be an* L_2 *frame for* $V_{j,p}$ *with frame constants independent from* j, p*. Moreover, assume that*

(J) $$\inf_{v \in V_{j,p}} \|f - v\|_{L_2(\mathbb{R})} \leq C C_J(p,s) 2^{-js} \|f\|_{H^s(\mathbb{R})}, \quad 0 \leq s < \gamma;$$

(B) $$\|v\|_{H^s(\mathbb{R})} \leq C C_B(p,s) 2^{js} \|v\|_{L_2(\mathbb{R})}, \quad v \in V_{j,p}, \quad 0 \leq s < \gamma.$$

Then for sufficiently small $\epsilon > 0$ *and* $\mathbf{a} \in \ell_1$,

$$\left\{ \frac{a_p}{C_B(p,s)^{1+\epsilon}} 2^{-js} \varphi_{p,j,k} : p, j \geq 0, k \in \mathbb{Z} \right\}$$

is a frame for $H^s(\mathbb{R})$, $0 < s < \gamma$.

Proof.

- Lower frame inequality (density) follows from (J)
- Upper frame inequality (stable synthesis): take $f = \sum_{p,j,k} c_{p,j,k} \frac{a_p}{C_B(p,s)^{1+\epsilon}} 2^{-js} \varphi_{p,j,k}$ and

$$\|f\|^2_{H^s(\mathbb{R})} = \sum_{p,p'} \frac{a_p a_{p'}}{C_B(p,s)^{1+\epsilon} C_B(p',s)^{1+\epsilon}} \sum_{j,k,j',k'} 2^{-s(j+j')} c_{p,j,k} c_{p',j',k'} \langle \varphi_{p,j,k}, \varphi_{p',j',k'} \rangle_{H^s(\mathbb{R})};$$

then use $\langle g, h\rangle_{H^s(\mathbb{R})} \leq \|g\|_{H^{s+\delta}(\mathbb{R})} \|h\|_{H^{s-\delta}(\mathbb{R})}$ for $\delta < \epsilon$, (J), (B) and the uniform frame property in $V_{j,p}$.

□

Direct estimates for Legendre quark spaces

Theorem

Let $\omega(f,t)_2 := \sup_{|h|\le t} \left\| f(\cdot + h) - f \right\|_{L_2(\mathbb{R})}$. *There exists* $C > 0$ *with*

$$\inf_{v \in V_{j,p}} \|f - v\|_{L_2(\mathbb{R})} \le C\omega\big(f, (p+1)^{-1} 2^{-j}\big)_2, \quad \text{for all } f \in L_2(\mathbb{R}), \quad j, p \ge 0.$$

Corollary

Let $0 < s \le 1$. *There exists* $C(s) > 0$ *with*

$$\inf_{v \in V_{j,p}} \|f - v\|_{L_2(\mathbb{R})} \le C(p+1)^{-s} 2^{-js} |f|_{H^s(\mathbb{R})}, \quad \text{for all } f \in H^s(\mathbb{R}), \quad j, p \ge 0.$$

Inverse estimates for Legendre quark spaces

Theorem

There exists $C > 0$, such that

$$\omega(v,t)_2 \leq C \min\left\{1, (p+1)^2 2^j t\right\}^{1/2} \|v\|_{L_2(\mathbb{R})}, \quad \textit{for all } v \in V_{j,p}, \quad j,p \geq 0.$$

Corollary

Let $0 < s < \frac{1}{2}$. There exists $C(s) > 0$ with

$$\|v\|_{H^s(\mathbb{R})} \leq C(p+1)^{2s} 2^{js} \|v\|_{L_2(\mathbb{R})}, \quad \textit{for all } v \in V_{j,p}, \quad j,p \geq 0.$$

Stability of Legendre quarks in $H^s(\mathbb{R})$, $s > 0$, revisited

Revisit dilated and translated Legendre quarks φ_p:

Corollary

For $0 < s < \frac{1}{2}$ *and sufficiently small* $\epsilon > 0$, *the system*

$$\left\{(p+1)^{-(1+2s+\epsilon)}\varphi_{p,j,k} : p, j \geq 0, k \in \mathbb{Z}\right\}$$

is a frame for $H^s(\mathbb{R})$.

- weights decay only algebraically in $p \to \infty$ (compare Triebel 1997ff.)

Monomial B-spline quarks

Another special case: monomial B-spline quarks

- $\varphi(x) := N_m(x)$, m-th order B-spline, $m \in \mathbb{N}$
- $\varphi_p(x) := (\frac{x}{m})^p \varphi(x)$, $p \geq 0$
- direct estimate for $V_{j,p}$:

$$\inf_{v \in V_{j,p}} \|f - v\|_{L_2(\mathbb{R})} \leq C(p+1)^{-s} 2^{-js} \|f\|_{H^s(\mathbb{R})}, \quad 0 \leq s \leq m$$

- inverse estimate for $V_{j,p}$:

$$\|f\|_{H^s(\mathbb{R})} \leq C(p+1)^{2s} 2^{js} \|f\|_{L_2(\mathbb{R})}, \quad \text{for all } f \in V_{j,p}, 0 \leq s < m - \frac{1}{2}$$

- in view of $\|\varphi_{p,j,k}\|_{L_2(\mathbb{R})} \eqsim (p+1)^{-(m-1/2)}$, the quark system $\{w_p 2^{-js} \varphi_{p,j,k} : p, j \geq 0, k \in \mathbb{Z}\}$ is a frame for $H^s(\mathbb{R})$ whenever $w_p^2 (p+1)^{2s-(2m-1)}$ is summable, e.g., $w_p = (p+1)^{m-2s-2-\epsilon}$.

Quarklets

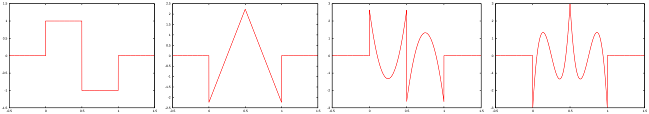
Issue: quarkonial systems are unstable in L_2

Way out: do some stabilization

- introduce (more) vanishing moments
- one out of many options: wavelet-type modification, e.g., of Legendre quarks

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad p \geq 0$$

- let's call them quarklets...



Quarklets: moment conditions

Legendre quarklets have vanishing moments:

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad \varphi_p(x) := \sqrt{2}L_p(2x-1)\varphi(x), \quad p \geq 0$$

Lemma

It holds that $\psi_p \perp \varphi_q(\cdot - k)$ for all $q \leq p$, $k \in \mathbb{Z}$. In particular, ψ_p has $p+1$ vanishing moments, $\psi_p \perp \mathbb{P}_p$.

This also works for monomial quarks and for other wavelet masks

$$\psi_p(x) := \sum_{k\in\mathbb{Z}} b_k \varphi_p(2x-k), \quad p \geq 0$$

Lemma

In this case, ψ_p has $\tilde{m}$ vanishing moments, like $\psi = \psi_0$.

Quarklets: stability in $L_2(\mathbb{R})$

$L_2(\mathbb{R})$ stability of Legendre quarklets:

Theorem

Let

$$\psi_{p,j,k} := \begin{cases} 2^{j/2}\psi_p(2^j \cdot -k), & j \geq 0 \\ \varphi_p(\cdot - k), & j = -1 \end{cases}, \quad \textit{for all } p \geq 0, j \geq -1, k \in \mathbb{Z}.$$

Then for each $\mathbf{a} \in \ell_1$,

$$\{a_p \psi_{p,j,k} : p \geq 0, j \geq -1, k \in \mathbb{Z}\}$$

is a frame for $L_2(\mathbb{R})$.

idea of the proof: orthonormality of $\{\psi_{p,j,k} : j \geq -1, k \in \mathbb{Z}\}$ for $p \geq 0$

(similar L_2 stability theorem holds for smoother monomial quarklets)

Quarklets: stability in $H^s(\mathbb{R})$, $s < 0$

For $\mathbf{a} \in \ell_1$, consider the $L_2(\mathbb{R})$ frame of Legendre quarklets

$$\{a_p \psi_{p,j,k} : p \geq 0, j \geq -1, k \in \mathbb{Z}\}$$

- Haar wavelet basis $\frac{1}{a_0}\Psi_H$ is *a* dual frame
- when rescaled, Ψ_H gets stable in $H^s(\mathbb{R})$, $0 \leq s < \frac{1}{2}$
- $\Rightarrow$ the rescaled quarklet system is a frame for $H^s(\mathbb{R})$, $-\frac{1}{2} < s < \frac{1}{2}$

(similar reasoning possible for smoother monomial quarklet frames)

Concluding Remarks

We have seen:

- subatomic decomposition of function spaces
- quarkonial frames of piecewise polynomial functions
- stability in $H^s(\mathbb{R})$, $0 < s < \gamma$
- wavelet-like modification yields "quarklets"
  - vanishing moments
  - stability in $H^s(\mathbb{R})$, $-\gamma < s < \gamma$

Work in progress:

- quark(let)s on bounded domains
- approximation spaces
- numerical applications
- ...

Thank you for you attention!