A class of Schur multipliers on some quasi-Banach spaces of infinite matrices

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Definition

1911 J. Schur introduced a new product among two matrices. If $A = (a_{jk})$ and $B = (b_{jk})$ are matrices of the same size (finite or infinite), their *Schur product* is defined to be the matrix of elementwise product

$$A * B = (a_{jk} b_{jk}).$$

G. Bennett in 1977 has written an influential paper where he studied the behavior, under Schur multiplication, of the norms $\| \cdot \|_{p,q}$, $1 \leq p, q \leq \infty$

$$\|A\|_{p,q} = \sup_{\|x\|_p \leq 1} \left(\sum_j \left| \sum_k a_{jk} x_k \right|^q \right)^{\frac{1}{q}}$$

.

He proved an important theorem about Schur-Toeplitz multipliers.

Definitions

A matrix A is a *Toeplitz matrix* if

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_{-1} & a_0 & a_1 & a_2 & \dots \\ a_{-2} & a_{-1} & a_0 & a_1 & \dots \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $(a_j)_{j=-\infty}^{\infty}$ is a sequence of complex numbers.

Let X, Y two quasi-Banach spaces of matrices. An infinite matrix A is called a *Schur multiplier* if $A * B \in Y$ for all $B \in X$.

Definition

The space of all Schur multipliers from X into Y equipped with the quasi-norm

$$||A||_{(X,Y)} = \sup_{||B||_Y \leq 1} ||A * B||_X$$

will be denoted by

$$(X, Y)$$

and becomes a quasi-Banach space.

Bennett's Theorem

A Toeplitz matrix A given by the sequence $(a_j)_{j=-\infty}^{\infty}$ as above is a Schur multiplier if and only if $\mu = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$ is a bounded Borel measure on $[0, 2\pi)$.

In his paper G. Bennett raised the following problem:

Characterize the Hankel matrices A which are Schur multipliers.

We recall that a matrix A is called a *Hankel matrix* if

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

1996 G. Pisier solved this problem:

Pisier's Theorem

A Hankel matrix A given by the sequence $(a_j)_{j=0}^{\infty}$ *is a Schur multiplier if and only if the Fourier multiplier* $\sum_n x_n e^{int} \to \sum_n a_n e^{int}$ *maps boundedly* $H^1(S_1)$ *into itself.*

Here $H^1(S_1)$ is the Hardy space of trace-class valued analytic functions and S_1 is the classical trace class of operators.

AP-Theorem

Finally 2002 Alexandrov and Peller characterized the Toeplitz-Schur multipliers from (S_p, S_p), $0 < p < 1$, proving

Let $0 < p < 1$. A Toeplitz matrix given by the complex sequence $(t_j)_{j=-\infty}^{\infty}$ belongs to (S_p, S_p) if and only if there exists a measure $\mu \in M_p$ such that $t_j = \widehat{\mu}(j)$, $j \in \mathbb{Z}$. Moreover, in this case

$$||T||_{(S_p,S_p)} = ||\mu||_{M_p},$$

where $M_p = \{\mu : \mathbb{T} \to \mathbb{C} \mid \mu = \sum_j \alpha_j \delta_{t_j},\ t_j \in \mathbb{T}, \}$ t_j distinct points , $||\mu||_{M_p} = (\sum_j |\alpha_j|^p)^{1/p} < \infty$ and δ_t is the Dirac measure concentrated at the point $t \in \mathbb{T}$.

The papers aforesaid showed that a complete description of all Schur multipliers, at least either for $B(\ell_2)$ or S_p, $0 < p \leq 1$, seems to be a difficult goal.
In this way it is natural to consider and study some interesting classes of Schur multipliers easier to characterize.
2005 Barza, Lie and myself introduced such a class of matrices given by a complex sequence $\alpha = (\alpha_j)_{j\geq 1}$. Namely we considered the matrix

$$[\alpha] = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \dots \\ \alpha_1 & \alpha_2 & \alpha_2 & \ddots \\ \alpha_1 & \alpha_2 & \alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and denote by *ms* the space of all sequence α such that $[\alpha] \in (B(\ell_2), B(\ell_2))$.

Remarks

We call the matrix $[\alpha]$ a *scalar matrix.* In the 2005 paper by using scalar matrices we extended to infinite matrices the well-known Haar's Theorem about the approximation of continuous functions by Haar polynomials.
We denoted by *pms* the space of all matrices of the form

$$\{\alpha\} = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \dots \\ 0 & \alpha_2 & \alpha_2 & \ddots \\ 0 & 0 & \alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

which belong to $(B(\ell_2), B(\ell_2))$.

Theorem BLP1

Let $b = (b_n)_{n\geq 1}$ *be a complex sequence.*
1) If $(i_n)_{n\geq 1}$ *is a strictly increasing sequence of natural numbers with* $i_1 = 0$*, then defining* $z_{i_n} = \max_{i_n < k \leq i_{n+1}} |b_k|$*, there is a constant* $R > 0$ *such that*

$$||\{b\}||_{(B(\ell_2),B(\ell_2))} \leq R \inf_{(i_n)_{n\geq 1}} \{||(z_{i_n})_{n\geq 1}||_2 + ||(z_{i_n} \log(i_{n+1} - i_n))_n||_\infty\}.$$

2) If $b \in pms$ *then*

$$\sup_{n\geq 1;\ p\geq 1} \frac{(\log n)^2}{n} \sum_{k=p}^{n+p} |b_k|^2 < \infty.$$

As an immediate consequence we have:

Corollary

$\ell_2 \subset ms \subset \ell_\infty$.

In the present talk we give some necessary and sufficient conditions in order that an upper (resp. a lower) triangular matrix A be a Schur multiplier on different quasi-Banach spaces of infinite matrices.
We start with the upper triangular matrices. Namely

Theorem (1)

1) Let Hardy space H^2, identified in an obviuos manner with a space of Toeplitz matrices. Then an upper triangular matrix $A = \{\alpha\}$, given by the sequence $\alpha = (\alpha_j)_{j \geq 0}$, belongs to $(H^2, B(\ell_2))$ if and only if $\alpha \in \ell_2$.
2) Let T_2 be the space of all upper triangular Hilbert-Schmidt matrices and $A = \{\alpha\}$. Then $A \in (T_2, B(\ell_2))$ if and only if $\alpha \in \ell_\infty$.

In order to prove this theorem we use an interesting formula due to V. Lie:

$$(1) \quad ||B||_{B(\ell_2)} = \sup_{||h||_2 \leq 1;\ h \in H_0^2[0,1]} \left(\sum_{k=1}^{\infty} \left| \int_0^1 \sum_{j=k}^{\infty} b_{kj} e^{2\pi ijt} h(-t) dt \right|^2 \right)^{1/2},$$

where B is an upper triangular matrix $B = (b_{kj})_{k \geq 1}^{\infty}$.

Next we use the important results of Bennett since 1996 in order to characterize the Schur multipliers of scalar type for some spaces of lower triangular infinite matrices belonging to Schatten classes S_p, $0 < p < \infty$. We denote these spaces by $\mathcal{L}\mathfrak{T}S_p$.
Unlike the case of matrices of S_p, $0 < p \leq 1$, whenever a general description of Schur multipliers of scalar type seems to be unknown, in case of triangular matrices this goal can be easily attained.

Definition

Let **f** be the space of all infinite sequences with only a finite number of non-zero elements. A norm Φ on **f** is called *symmetric* if $\Phi(a) = \Phi(a^*)$, for all $a \in \mathbf{f}$, that is if Φ is invariant to permutations and to applications $a_n \to e^{i\theta_n} a_n$. Here $a^* = (a_n^*)_{n=1}^{\infty}$ is the decreasing rearrangement of the sequence $a_n \to 0$ as $n \to \infty$.
We say that the sequence $(a_n)_n$ belongs to the space s_Φ, if and only if there exists $\lim_{n\to\infty} \Phi(a_1, \ldots, a_n, 0, 0, \ldots) = \Phi(a)$ and it is finite.
We note by S_Φ the space of all compact operators A on ℓ_2 with the sequence of their singular numbers $(\mu_n(A))$ belonging to s_Φ.
For $A \in S_\Phi$ we put $\Phi(A) = \Phi((\mu_n(A))_n)$.

The following theorem is well-known:

Theorem AH

Theorem

Let Φ_1, Φ_2, Φ_3 be symmetric norms on sequence spaces such that if $a \in s_{\Phi_2}, b \in s_{\Phi_3}$ then ab, given by $(ab)_n = a_n b_n$, belongs to s_{Φ_1} and

$$\Phi_1(ab) \le \Phi_2(a)\Phi_3(b).$$

Then, if $A \in S_{\Phi_2}$, $B \in S_{\Phi_3}$, it follows that $AB \in S_{\Phi_1}$ and

$$\Phi_1(AB) \le \Phi_2(A)\Phi_3(B).$$

Since $\ell_p \cdot \ell_q = \ell_r$, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$; $0 < p, q, r \leq \infty$:

Corollary

1) Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$; $0 < p, q, r \leq \infty$. Then $[\alpha] \in (S_p, S_r)$ if and only if $\alpha \in \ell_q$.
2) Let $w_n \searrow 0$ a sequence and $0 < p, q \leq \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Since, by classical Hoelder's inequality, $\ell_{p,w} \cdot \ell_{q,w} = \ell_{1,w}$, we have $[\alpha] \in (S_{p,w}, S_{1,w})$ if and only if $\alpha \in \ell_{q,w}$, where
$\ell_{p,w} = \{x = (x_n)_n | \ (\sum_{n=1}^{\infty} (x_n^*)^p w_n)^{1/p} = ||x||_{p,w} < \infty\}$.

Definition

We call *the Bergman-Schatten space of order p*, $0 < p < \infty$ and we note by $L_a^p(\ell_2)$ the space of all upper triangular matrices A such that $||A||_{L_a^p(\ell_2)} = \left(\int_0^1 \left\|\sum_{k=0}^{\infty} A_k r^k\right\|_{S_p}^p 2rdr\right)^{1/p} < \infty$.

View Hoelder's inequality we get:

Theorem

Let $1 \leq p < \infty$. Then $[\alpha] \in \left(L_a^p(\ell_2), L_a^1(\ell_2)\right)$ if and only if $\alpha \in \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Using the beautiful results of Bennett since 1996 we can describe the Schur multipliers of scalar type also for others quasi-Banach spaces of matrices.
First we recall some definitions:

Definition

Let $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \dots)$ a sequence with positive terms and let's assume that $a_1 > 0$. Let $A_n = a_1 + a_2 + \cdots + a_n$. For $p > 0$ we define

$$d(\mathbf{a}, p) = \{\mathbf{x} : \sum_{n=1}^{\infty} a_n \sup_{k \geq n} |x_k|^p < \infty\}$$

and

$$g(\mathbf{a}, p) = \{\mathbf{x} : \sum_{k=1}^{n} |x_k|^p = \mathcal{O}(A_n)\}.$$

definitions

We note by

$$||\mathbf{x}||_{d(\mathbf{a},p)} = \left(\sum_{n=1}^{\infty} a_n \sup_{k\geq n} |x_k|^p\right)^{1/p}$$

and

$$||\mathbf{x}||_{g(\mathbf{a},p)} = \sup_n \left(\frac{1}{A_n}\sum_{k=1}^{n} |x_k|^p\right)^{1/p}.$$

Finally, for $p > 1$, let

$$ces(p) = \{\mathbf{x} : \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p} = ||\mathbf{x}||_{ces(p)} < \infty\}.$$

We note now by $d(\mathbf{a}, p; \infty)$, $g(\mathbf{a}, p; \infty)$, $ces(p; \infty)$ and $\ell_{p;\infty}$ the spaces of upper triangular infinite matrices $A = \sum_{k=0}^{\infty} A_k$, that have on the kth diagonal the sequence $(a_k^l)_{l=1}^{\infty} \in d(\mathbf{a}, p)$ (respectively $g(\mathbf{a}, p)$, $ces(p)$, ℓ_p,) and $||A|| = \sup_k ||(a_k^l)_l||_{d(\mathbf{a},p)} < \infty$ and similarly for $g(\mathbf{a}, p)$, $ces(p,)$ ℓ_p.

Theorems

Theorem

1) Let $1 < p < \infty$. *Then* $\alpha \in g(p^*)$, *where* $\frac{1}{p} + \frac{1}{p^*} = 1$, *if and only if* $[\alpha] \in (\ell_{p,\infty}, ces(p;\infty))$, *where* $g(p^*) = g(\mathbf{a}, p^*)$, *with* $\mathbf{a} = (1, 1, \dots)$.
2) Let $0 < p < \infty$. *Then* $[\alpha] \in (d(\mathbf{a}, p;\infty), \ell_{p,\infty})$ *if and only if* $\alpha \in g(\mathbf{a}, p)$.

Similarly, noting by $d(\mathbf{a}, p; q)$, $g(\mathbf{a}, p; q)$, $ces(p; q)$, $\ell_{p,q}$, $0 < q < \infty$, the spaces of upper triangular infinite matrices having the sequences on the diagonals belonging to $d(\mathbf{a}, p)$, $g(\mathbf{a}, p)$ etc. and $||A||_{d(\mathbf{a},p;q)} = \left(\sum_{k=0}^{\infty} ||A_k||^q_{d(\mathbf{a},p)}\right)^{1/q}$ etc., it follows:

Theorem

Theorem

1) For $1 < p < \infty$, $[\alpha] \in (\ell_{p,q}, ces(p;q))$ *if and only if* $\alpha \in g(p^*)$.
2) For $0 < p < \infty$, $[\alpha] \in (d(\mathbf{a}, p; q), \ell_{p;q})$ *if and only if* $\alpha \in g(\mathbf{a}, p)$.