Quarkonial Decompositions and hp-Type Partition of Unity Methods

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Contents:

1) Atomic and subatomic decompositions for analysis and numerical methods
2) Multilevel partition of unity methods (MPUM) and hp-type approximation
3) Alternative proof of frame property for quarkonial systems in Besov spaces, with slower weight decay, in special cases
4) Open questions and invitation to contribute!

Joint work with S. Dahlke (Marburg) and T. Raasch (Mainz)

Atoms for Besov spaces

(p,s) – atoms $(s > \sigma_p \equiv d(1/p-1)_+)$: Q - dyadic cube

(i) $\text{supp } a_Q \subseteq c_0 Q$

(ii) $|D^\gamma a_Q| \le c_1 |Q|^{-(1/p+|\gamma|/d)}, \quad |\gamma| \le [s]+1$

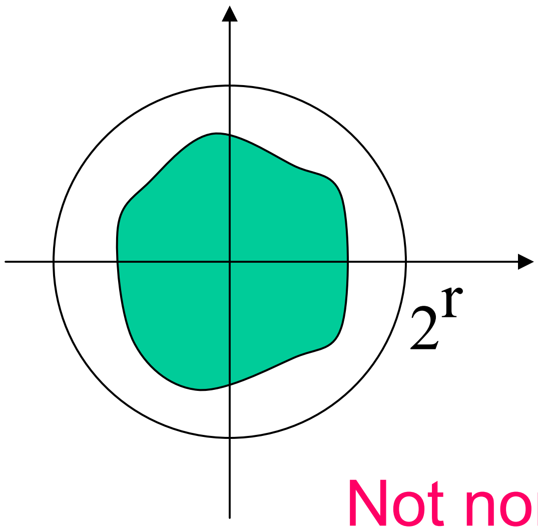
Normalized in L_p

$$f \in B^s_{p,p} \quad \Leftrightarrow \quad (*) \quad f = \sum_{j\ge 0} \sum_{Q \in \mathcal{Q}_j} c_Q a_Q$$

$$\| f \|_{B^s_{p,p}} \approx ||| f |||_{B^s_{p,p}} = \inf_{(*)} \left(\sum_{j\ge 0} \sum_{Q \in \mathcal{Q}_j} 2^{jsp} |c_Q|^p \right)^{1/p}$$

Quarkonial decompositions

Introduced by Triebel (1997+) for the study of function spaces and differential operators on fractal sets and quasi-metric spaces. Lattice version:



$$\varphi \in C^\infty, \quad \operatorname{supp} \varphi \subset B(0,2^r)$$

$$\sum_{\alpha \in Z^d} \varphi(x - \alpha) = 1$$

$$\varphi_\beta(x) = (2^{-r} x)^\beta \varphi(x)$$

Not normalized in L_p

$$\phi_\lambda(x) \equiv 2^{jd/p} \varphi_\beta(2^j x - \alpha), \quad \lambda \equiv (j, \alpha, \beta).$$

(Modification: PU functions centered not at lattice $2^{-j}Z^d$ but on quasi-uniform grid of step-size ~ 2^{-j} of level j.)

Theorem (Triebel, 1997+). If $s > \sigma_p$, then

$$f \in B^s_{p,p} \quad \Leftrightarrow \quad (*) \quad f = \sum_\lambda c_\lambda \phi_\lambda$$

$$\| f \|_{B^s_{p,p}} \approx ||| f |||_{B^s_{p,p}} := \inf_{(*)} \left(\sum_\lambda 2^{jsp} | c_\lambda |^p \right)^{1/p}$$

The norms $||| f |||_{B^s_{p,p}}$ and $\| f \|_{B^s_{p,p}}$ are equivalent, with constants depending on s and p. Moreover, for $p = 2$ the system

$$\Phi = \{2^{-js} \phi_\lambda\}$$

is a frame in the Sobolev space H^s . Results are also available for (weighted) spaces on domains.

Multilevel Partition of Unity Methods

Mesh-less discretization methods provide an alternative to the high cost of domain partitioning and building smooth ansatz functions in finite element methods. Also motivated by particle methods.

Triangulation ⟷ Point cloud

Appear under different names, with small variations:

- Smoothed particle hydrodynamics (Monaghan et al.)
- Moving least-squares methods (Lancaster, Salkauskas)
- Element free Galerkin methods (Belytschko, Lu)
- Reproducing kernel particle methods (Liu et al.)
- Generalized finite element methods (Babuska et al.)
- hp – clouds (Duarte, Oden)
- Partition of unity methods (Babuska, Melenk)

Ingredients of the PUM

1) Point distribution $\longrightarrow$ Overlapping cover of domain $\longrightarrow$ "Hill" functions $\longrightarrow$ Partition of unity functions

$$x_i \rightarrow \omega_i \rightarrow \varphi_i(x) \rightarrow \phi_i(x) = \frac{\varphi_i(x)}{\sum_\ell \varphi_\ell(x)}$$

2) Local enrichment: "Modulate" the PU functions, e.g.,

$$\phi_{i,\beta}(x) = (x - x_i)^\beta \phi_i(x), \quad |\beta| \le p_i.$$

3) Global ansatz function space:

$$V_{PUM} = \text{span}\,\{\phi_{i,\beta}(x)\}$$

Plus: Very flexible.
Smoothness at no cost.

Minus: Bases? Essential b.c.?
Numerical integration costly.

Multilevel PUM

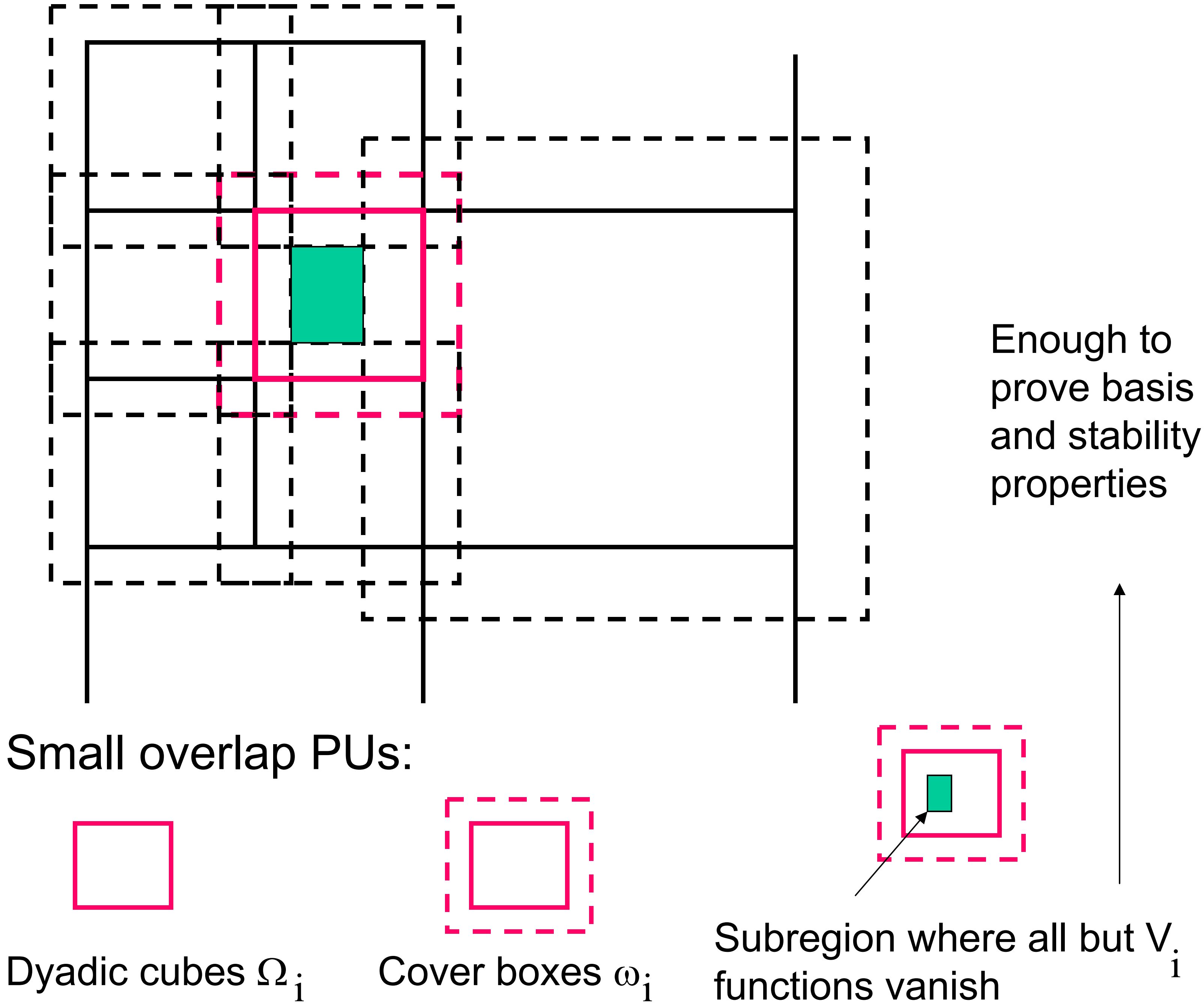
Introduced by Griebel and Schweitzer (2000+). Creates hierarchical, tree-structured cover system from initially given point set, and extracts PUM spaces of various levels by tree pruning.

Main features:

PUs and PUM spaces on each level are constructed independently of each other, preserve all good (and bad) properties of the PUM but often lead to non-nested MRA:

$$V_0 \not\subset V_1 \not\subset \cdots \not\subset V_j \not\subset \cdots \not\subset V_J$$

Support for adaptivity and parallelization, multilevel structure can be used to full extent.


Enough to prove basis and stability properties
Small overlap PUs:
Dyadic cubes Ω_i
Cover boxes ω_i
Subregion where all but V_i functions vanish

Hierarchical "Frames" by Weighting

Quarkonial systems are hierarchical with respect to the "modulation" degree, i.e.,

$$\Phi^k = \{q_\lambda : \ |\beta| \leq k\} = \Phi^{k-1} \cup \{q_\lambda : \ |\beta| = k\},$$

Normalized in L_p

$$q_\lambda(x) = w_\lambda(x - x_{j,i})^\beta \phi_{j,i}(x), \ \lambda = (j, i, \beta).$$

and very redundant representation systems in Besov spaces of finite smoothness: For finite $s > \sigma_p$, there is often a finite k such that already Φ^k is stable:

$$\|f\|_{B^s_{p,p}} \approx |||f|||_k := \inf \Big(\sum_{\lambda:|\beta|\leq k} 2^{jsp}|c_\lambda|^p\Big)^{1/p}$$

Constants depend on k (how much?)

Consequence: Stability of the full quarkonial system can be achieved if we “penalize” the influence of additional

$$q_\lambda(x), \quad |\beta| > k,$$

by weighting them with factors << 1.

In some sense, this is what is behind Triebel’s stability result: His quarks are scaled such that

$$\exists \delta < 1 : \quad \|\phi_\lambda\|_{L_p} = \mathrm{O}(\delta^{|\beta|}), \quad |\beta| \to \infty.$$

Such exponential decay of weight factors is, from a numerical point of view, not desirable, as we then tend to over-penalize the use of quarks with high degree, and loose some of the potential advantages of a hp-type scheme. Is algebraic decay sufficient?

Results

1) Abstract theory for proving stability with certain weights based on
   (J) Jackson-type estimates for some k
   (B) Bernstein-type estimates for all k
   (L) Lower L_p stability bound for some k
   (U) Upper L_p stability bound for all k
   Appropriate weights can be computed from the constants in the (B) and (U) bounds.

Example: (B)

$$\|v_j^k\|_{B^t_{p,p}} \leq C A_{k,t} 2^{jt} \|v_j^k\|_{L_p}, \quad \sigma_p < t < s_p$$

$$v_j^k = \sum_{|\beta| \leq k} \sum_i c_\lambda q_\lambda$$

Results

2) Verification of (J), (B), (L), (U) under additional constraints on the PUs, with emphasis on best possible constants in (U) (easy) and (B) (harder).
Stability with algebraically decaying weights established for refinable PUs consisting of piecewise polynomials

Example: For dimension d=1, take PUs obtained from B-splines of some fixed degree and smoothness on nested quasi-uniform partitions of mesh-width $\approx 2^{-j}$. Then

$$\|f\|_{B^t_{p,p}} \approx \inf_{(*)} \Big(\sum_\lambda 2^{jsp} (|\beta|+1)^{\alpha p} |c_\lambda|^p\Big)^{1/p},$$

$$(*) \quad f = \sum_\lambda c_\lambda q_\lambda, \qquad \|q_\lambda\|_{L_p} \approx 1,$$

$$\alpha > 2s+1-1/p,\ 0 < s < R+1+1/p,\ 1 \le p < \infty.$$

Discussion

- For technical details, see forthcoming HIM Bonn preprint (or ask D/O/R).

- Challenge: For PUs, generated by the dyadic shifts and translates of a fixed sufficiently smooth refinable function, does there a Bernstein estimate of the form

$$\|v_j^k\|_{B^t_{p,p}} \leq C(k+1)^{2t} 2^{jt} \|v_j^k\|_{L_p}, \quad \sigma_p < t < s_\phi$$

  hold? The case p=2 is of particular interest.

- Optimal weights? Nonlinear n-term approximation?

- For some other MPUM related issues of quarkonial systems (moment conditions, negative smoothness), see Thorsten Raasch's talk.

Motivation: hp Theory !

The approximation theory of the hp-method is ingenious work by Babuska and co-workers but painful to look at.

High-level summary:

1) hp finite element spaces lead to exponential approximation rates

$$\inf \| u - u_n \|_{H^k} \leq C \exp(-cn^{\alpha}), \quad \alpha = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{5} \; (d = 1,2,3)$$

provided the function u is "piecewise-analytic" with some isolated singularities (belongs to a certain countably normed weighted Besov space for p=2) .

2) Solutions to standard elliptic PDE and IE problems can be shown to belong to the latter space.

It is intriguing to ask, whether these countably normed spaces can be replaced by spaces which

a) admit a characterization in terms of (anisotropic) quarkonial decompositions,

b) will lead to nonlinear n-term rates for tree approximation of exponential type, and

c) carry a kind of elliptic regularity theory that includes the typical applications (solutions with point and edge singularities)

We believe that our results are a first step in this direction.