Natasha Samko

On the weighted boundedness of the singular integral operator in Morrey spaces

FSDONA-2011

Tabarz, September, 2011

1 Introduction

We consider the classical Morrey space with weight:

$$L^{p,\lambda}(\Omega, w) := \{f : \|f\|_{p,\lambda;w} < \infty\}, \quad 1 \le p < \infty, \ 0 \le \lambda \le 1,$$

where $\Omega \subseteq \mathbb{R}^n$, defined by the norm

$$\|f\|_{p,\lambda;w} := \sup_{x,r} \left(\frac{1}{|B(x,r)|^\lambda} \int\limits_{B(x,r)} |f(y)|^p w(y)\, dy \right)^{\frac{1}{p}},$$

where we always assume that a function f is continues beyond Ω as zero whenever necessary. We write $\|f\|_{p,\lambda}$ in the non-weighted case $w \equiv 1$.

We have studied the Hardy, Singular, Maximal and Potential operators in such spaces and have obtained sufficient and necessary conditions for their boundedness under some restrictions on the weight functions.

As shown in [48, N. Samko], the Muckenhoupt class A_p may not be an appropriate class of weights for the case of Morrey spaces. The appropriate ”Muckenhoupt-type” class for the Morrey spaces must depend on the parameter λ.

As proved in [48] for the one-dimensional case, the singular integral operator

$$Sf(x) = \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(t)\,dt}{t-x}$$

is bounded in the space $L^{p,\lambda}(\mathbb{R}^1, w)$ **with the power weight** $w(x) = |x-a|^\nu,\ a \in \Omega$, **if and only if**

$$\lambda - 1 < \nu < \lambda + p - 1 \tag{1.1}$$

which is a shifted interval in comparison with the Muckenhoupt condition

$$-1 < \nu < p - 1.$$

Thus, condition (1.1) **partially deletes Muckenhoupt power weights, but on the other hand, adds new ones.**

As is known, a description of all the admissible weights for Morrey spaces, similar to the Muckenhoupt class A_p **is an open problem.**

Since the A_p**-condition**

$$A_p: \qquad \sup_B \left(\frac{1}{|B|} \int\limits_B w(x)\,dx \right) \left(\frac{1}{|B|} \int\limits_B w^{1-p'}(y)\,dy \right)^{p-1} < \infty \tag{1.2}$$

has the form

$$\sup_B \frac{1}{|B|} \|v\|_{L^p(B)} \left\| \frac{1}{v} \right\|_{L^{p'}(B)} < \infty, \qquad v = w^{\frac{1}{p}}, \tag{1.3}$$

where $\sup$ **is taken with respect to all balls in** $\mathbb{R}^n$ **one can expect that the corresponding Muckenhoupt-type class** $A_{p,\lambda}$ **may be**

defined by the condition

$$A_{p,\lambda}: \qquad \sup_B \frac{1}{|B|}\|v\|_{L^{p,\lambda}(B)}\left\|\frac{1}{v}\right\|_{[L^{p,\lambda}]'(B)} < \infty, \quad v = w^{\frac{1}{p}}, \tag{1.4}$$

where $[L^{p,\lambda}]'$ may stand for the dual (or predual ?) of the Morrey space. The preduals of Morrey spaces were studied in [23], [25], [34] and [54]. Their characterizations are known to be given in capacitory terms and/or in terms of the so called (q, λ)-atomic decompositions, which makes them uneasy in concrete applications.

We introduce a certain class $\mathcal{A}_{p,\lambda} = \mathcal{A}_{p,\lambda}(\mathbb{R}^n)$ of weights, which might be conditionally called a *pre-Muckenhoupt class for Morrey spaces*. It turns into the Muckenhoupt class A_p when $\lambda = 0$ and we show that the belongness of a weight to this class is necessary for the one-dimensional singular integral operator (Hilbert transform) to be bounded in the Morrey space.

2 Main result

We start with some *à priori assumptions and the class* $\mathcal{A}_{p,\lambda}$.

The definition

$$A_p: \qquad \sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w^{1-p'}(y)\,dy\right)^{p-1} < \infty$$

of the Muckenhoupt class A_p for the spaces L^p (the case $\lambda = 0$) preassumes that

$$\textbf{the functions} \quad w \quad \textbf{and} \quad w^{1-p'} \quad \textbf{are locally in} \quad L^p. \tag{2.1}$$

What should be similar *à priori* assumptions for the Morrey spaces?

In the case of the power weight $w = |x-a|^\nu$ **the conditions in** (2.1) **mean that**

$$\nu > -n \quad \textbf{and} \quad \nu < n(p-1),$$

respectively.

In the case of Morrey spaces, the corresponding interval

$$(-n,\ n(p-1)\ ) \quad \textbf{should be shifted to} \quad (n\lambda - n,\ n\lambda + n(p-1)\ ),$$

as noted in

$$\lambda - 1 < \nu < \lambda + p - 1$$

in the one-dimensional case $n = 1$**.**

Thus for general weights we expect that the *à priori* assumption

$w \in L^1_{\text{loc}}$ **must be replaced by some more restrictive condition,**

while the condition

$w^{1-p'} \in L^1_{\text{loc}}$ **is expected to be weakened,**

both in dependence on the parameter λ.

As a substitution of the first assumption

$$w \in L^1_{\text{loc}}$$

we will use now the following natural condition on the weight w**:**

$$\chi_B \in L^{p,\lambda}(\Omega, w) \quad \Longleftrightarrow \quad \chi_B w^{\frac{1}{p}} \in L^{p,\lambda}(\Omega) \tag{2.2}$$

for all the balls B, **where** χ_E **denotes the characteristic function of an open set** $E \subset \Omega$. **As a substitution of another condition**

$$w^{1-p'} \in L^1_{\text{loc}}$$

we introduce the condition

$$\chi_B \in L^{p,\lambda}\left(\Omega, w^{-\frac{1-\lambda}{\lambda+p-1}}\right) \quad \Longleftrightarrow \quad \chi_B w^{-\frac{1}{\lambda+p-1}} \in L^{p,\lambda}(\Omega, w)\,, \tag{2.3}$$

which turns into $w^{1-p'} \in L^1_{\text{loc}}$ when $\lambda = 0$. With the notation $w(E) := \int\limits_E w(x)\,dx$, the conditions (2.2) and (2.3) have the form

$$\sup_B \frac{w(B \cap B(x_0,r))}{|B|^\lambda} < \infty, \quad \sup_B \frac{w^{-\frac{1-\lambda}{\lambda+p-1}}(B \cap B(x_0,r))}{|B|^\lambda} < \infty, \tag{2.4}$$

respectively, where the $\sup$ is taken with respect to all balls $B \subset \Omega$.

Definition 2.1. A weight function w is called (p,λ)-*admissible weight*, if it satisfies the assumptions (2.2)-(2.3).

The condition (2.2) of belongness of functions χ_B to the weighted space $L^{p,\lambda}$ is quite natural. As for the exponent $-\frac{1-\lambda}{\lambda+p-1}$ in the condition (2.3), its choice originated in particular from the upper bound in the conditions $\lambda - 1 < \nu < \lambda + p - 1,$ known to be necessary and sufficient for power weights.

Now we introduce the class $\mathcal{A}_{p,\lambda}$ by the following definition.

Definition 2.2. By $\mathcal{A}_{p,\lambda}$ we denote the class of (p,λ)-admissible weights satisfying the condition

$$\mathcal{A}_{p,\lambda}: \qquad \sup_B \frac{\|\chi_B\|_{p,\lambda;w}}{\|\chi_B\|_{p,\lambda;w_*}} \left(\frac{1}{|B|}\int\limits_B w^{-\frac{1}{\lambda+p-1}} dy\right) < \infty, \quad w_* = w^{-\frac{1-\lambda}{\lambda+p-1}}, \tag{2.5}$$

where $\sup$ is taken with respect to all balls. Obviously we obtain the Muckenhoupt class A_p when $\lambda = 0$.

$n = 1$

We find it convenient to use the notation

$$I = I(x, r) = \{y : x - r < y < x + r\}$$

for the one-dimensional balls. We assume that the weight w **is** (p, λ)**-admissible in the sense of Definition 2.1 which now means that**

$$\chi_I \in L^{p,\lambda}(\mathbb{R}, w) \tag{2.6}$$

and

$$\chi_I \in L^{p,\lambda}\left(\mathbb{R}, w^{-\frac{1-\lambda}{\lambda+p-1}}\right), \tag{2.7}$$

for all intervals $I \subset \mathbb{R}$.

Suppose that the singular operator S **is bounded in the weighted Morrey space:**

$$\|Sf\|_{p,\lambda;w} \leq k\|f\|_{p,\lambda;w}. \tag{2.8}$$

with a (p, λ)**-admissible weight** w**.**

Theorem 2.3. ***Let the assumption*** (2.8) ***hold with a*** (p, λ)***-admissible weight*** w***. Then***

$$\sup_{I:|I|\leq 1} \frac{\|\chi_I\|_{p,\lambda;w}}{\|\chi_I\|_{p,\lambda;w_*}} \left(\frac{1}{|I|} \int_I w^{-\frac{1}{\lambda+p-1}} dy \right) \leq 2k < \infty, \quad w_* = w^{-\frac{1-\lambda}{\lambda+p-1}} \tag{2.9}$$

with $k = \|S\|_{L^{p,\lambda}(\mathbb{R},w)\to L^{p,\lambda}(\mathbb{R},w)}$.

Corollary 2.4. ***Let*** w ***be a*** (p, λ)***-admissible weight. The condition*** $w \in \mathcal{A}_{p,\lambda}$ ***is necessary for the boundedness of the singular operator in the weighted Morrey space*** $L^{p,\lambda}(\mathbb{R}, w)$.

3 Norms of characteristic functions of balls in Morrey spaces

Every simple function belongs to non-weighted Morrey spaces, while it is not the case in general for weighted Morrey spaces. any such belongness for functions χ_B imposes conditions on the weight, which were already discussed below in some *à priori assumptions and the class* $\mathcal{A}_{p,\lambda}$. The aim of this section is to shed more light on such belongness and to give some estimations of the norms $\|\chi_B\|_{p,\lambda;w}$ involved in the $\mathcal{A}_{p,\lambda}$-condition (2.5) in the case $\Omega = \mathbb{R}^n$.

3.1 The non-weighted case

Let $B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$. Fix a ball $B(x_0,r_0)$. The following lemma holds.

Lemma 3.1. ***Let*** $1 \le p < \infty, \;\; 0 \le \lambda \le 1$. ***The formula***

$$\left\|\chi_{B(x_0,r_0)}\right\|_{p,\lambda} = |B(x_0,r_0)|^{\frac{n(1-\lambda)}{p}} = (\omega_n r_0^n)^{\frac{1-\lambda}{p}} . \qquad (3.1)$$

is valid, where $\omega_n = |\mathbb{S}^{n-1}|$.

3.2 The weighted case

In the weighted case we cannot already write a precise formula of type (3.1) localized to the point x_0, since the values of the weight w at the points x different from x_0 may already heavily influence on the value of the norm $\left\|\chi_{B(x_0,r_0)}\right\|_{p,\lambda;w}$.

With the usual notation $w(E) = \int_E w(x)\,dx$ we can write

$$\left\|\chi_{B(x_0,r_0)}\right\|_{p,\lambda;w} = \sup_{\substack{x\in\mathbb{R}^n, r>0: \\ |x-x_0|<r+r_0}} \left(\frac{w(B(x,r)\cap B(x_0,r_0))}{(\omega_n r^n)^\lambda}\right)^{\frac{1}{p}}, \qquad (3.2)$$

where we took into account that $w(B(x,r)\cap B(x_0,r_0) = \emptyset$ when $|x-x_0| > r+r_0$; however, (3.2) **is just a direct usage of the definition of the norm. From** (3.2) **we can derive the following statement.**

Lemma 3.2. ***The norm*** $\left\|\chi_{B(x_0,r_0)}\right\|_{p,\lambda;w}$ ***admits the estimate***

$$\frac{1}{\omega_n^\lambda}\sup_{0<r<r_0}\left(\frac{w(B(x_0,r)}{r^{n\lambda}}\right)^{\frac{1}{p}} \le \left\|\chi_{B(x_0,r_0)}\right\|_{p,\lambda;w} \le \frac{1}{\omega_n^\lambda}\sup_{\substack{|x-x_0|<2r_0\\0<r<r_0}}\left(\frac{w(B(x,r)}{r^{n\lambda}}\right)^{\frac{1}{p}}. \tag{3.3}$$

The following corollary provides conditions on the weight function w **for which simple functions belong to the weighted space** $L^{p,\lambda}(\mathbb{R}^n,w)$.

Corollary 3.3. ***For the characteristic function*** $\chi_{B(x_0,r_0)}$ ***of a ball*** $B(x_0,r_0)$ ***to belong to the space*** $L^{p,\lambda}(\mathbb{R}^n,w)$***, the condition***

$$\sup_{\substack{|x-x_0|<2r_0\\0<r<r_0}}\frac{w(B(x,r))}{r^{n\lambda}} < \infty \tag{3.4}$$

is sufficient, and the condition

$$\sup_{0<r<r_0}\frac{w(B(x_0,r))}{r^{n\lambda}} < \infty \tag{3.5}$$

is necessary.

4

Various sufficient conditions for some classes of weights have been also obtained and for the case of more general spaces

$\mathcal{L}^{p,\varphi}(\Omega, w)$ **defined by**

$$\sup_{\substack{x\in\mathbb{R}^n,\\ r>0}} \frac{1}{\varphi(r)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^p \, w(y)dy < \infty, \tag{4.6}$$

We consider weights with non-exponential behaviour at the origin and infinity. By V$_{\pm}$, we denote the classes of functions w **which are non-negative on** $[0, \infty]$ **and positive on** $(0, \infty)$, $0 < \ell \leq \infty$, **defined by the following conditions:**

$$\mathbf{V}_+ : \qquad \frac{|w(x) - w(y)|}{|x - y|} \leq C \frac{w(x_+)}{x_+}, \tag{4.7}$$

$$\mathbf{V}_- : \qquad \frac{|w(x) - w(y)|}{|x - y|} \leq C \frac{w(x_-)}{x_+}, \tag{4.8}$$

where $x, y \in (0, \infty), x \neq y$, **and**

$$x_+ = \max(x, y), \quad x_- = \min(x, y).$$

Theorem 4.4. ***Let*** $1 < p < \infty$ ***and*** $\varphi(r)$ ***a non-negative measurable function satisfying the conditions***

$$\varphi(r) \geq cr \quad \textit{and} \quad \int_r^{\infty} \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{1}{p}+1}} dt \leq C$$

and let the weight function $w \in \mathbf{V}_- \cup \mathbf{V}_+$ ***satisfy the conditions***

$$\int\limits_{\varepsilon}^{\infty} \frac{t^{-\frac{1}{p}-1} \varphi^{1/p}(t)}{w^{\frac{1}{p}}(t)} \, dt < \infty.$$

Then the singular operator

$$Sf(x) = \int\limits_{\mathbb{R}} \frac{f(y)\,dy}{y - x}$$

is bounded in the weighted generalized Morrey space $\mathcal{L}^{p,\varphi}(\mathbb{R},w)$ ***under the conditions***

$$\sup_{x\in\Omega,r>0}\frac{1}{\varphi(r)}\int_{B(x,r)} w(|y|)|y|^{-p}\left(\int_0^{|y|}\frac{t^{\frac{1}{p'}-1}\varphi^{\frac{1}{p}}(t)}{w^{\frac{1}{p}}(t)}dt\right)^p dy<\infty$$

and

$$\sup_{\substack{x\in\mathbb{R},\\ r>0}}\frac{1}{\varphi(r)}\int_{B(x,r)}\left(\int_{|y|}^{\infty} t^{-\frac{1}{p}-1}\varphi^{\frac{1}{p}}(t)dt\right)^p dy<\infty, \tag{4.9}$$

in the case $w\in\mathbf{V}_+$ ***and conditions***

$$\sup_{\substack{x\in\mathbb{R},\\ r>0}}\frac{1}{\varphi(r)}\int_{B(x,r)}|y|^{-p}\left(\int_0^{|y|} t^{\frac{1}{p'}-1}\varphi^{\frac{1}{p}}(t)dt\right)^p dy<\infty, \tag{4.10}$$

and

$$\sup_{\substack{x\in\mathbb{R},\\ r>0}}\frac{1}{\varphi(r)}\int_{B(x,r)} w(|y|)\left(\int_{|y|}^{\infty}\frac{t^{-\frac{1}{p}-1}\varphi^{\frac{1}{p}}(t)}{w^{\frac{1}{p}}(t)}dt\right)^p dy<\infty, \tag{4.11}$$

in the case $w\in\mathbf{V}_-$.

sketch of the proof:

We prove that the following pointwise estimates hold

$$\left|\left(\varrho S\frac{1}{\varrho}-S\right)f(x)\right|\le C\frac{\varphi(x)}{x}\int\limits_0^x\frac{|f(t)|dt}{\varphi(t)}+C\int\limits_x^{\ell}\frac{|f(t)|}{t}dt, \tag{4.12}$$

when $\varphi\in\mathbf{V}_+$**, and**

$$\left|\left(\varrho S\frac{1}{\varrho}-S\right)f(x)\right|\le\frac{C}{x}\int\limits_0^x|f(t)|dt+C\varphi(x)\int\limits_x^{\ell}\frac{|f(t)|dt}{t\varphi(t)}, \tag{4.13}$$

when $\varphi\in\mathbf{V}_-$.

References

[1] C. Bennett and R. Sharpley. *Interpolation of operators.*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[2] V.I. Burenkov and H. Guliyev. Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces. *Studia Math.*, 163(2):157–176, 2004.

[3] J. Duoandikoetxea. *Fourier Analysis.* Amer. Math. Soc., ”Graduate Studies”, vol. 29, 2001.

[4] G.T. Dzhumakaeva and K. Zh. Nauryzbaev. Lebesgue-Morrey spaces. *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.*, (5):7–12, 79, 1982.

[5] V. Guliev. *Integral operators on function spaces on homogeneous groups and on domains in R^n (in Russian)*. PhD thesis, Moscow, Steklov Math. Inst., 1994.

[6] V. Guliev. *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications.* Baku, 1999. in Russian.

[7] V. Guliev, J. Hasanov, and S. Samko. Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. *Math. Scand.*, 107:285–304, 2010.

[8] J.L. Journé. *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, volume 994 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[9] Lech Maligranda. Indices and interpolation. *Dissertationes Math. (Rozprawy Mat.)*, 234:49, 1985.

[10] Lech Maligranda. *Orlicz spaces and interpolation*. Departamento de Matemática, Universidade Estadual de Campinas, 1989. Campinas SP Brazil.

[11] W. Matuszewska and W. Orlicz. On some classes of functions with regard to their orders of growth. *Studia Math.*, 26:11–24, 1965.

[12] T. Mizuhara. Boundedness of some classical operators on generalized Morrey spaces. In S.Igari, editor, *Harmonic Analysis*, pages 183–189. Springer, 1991. ICM 90 Satellite Proceddings.

[13] C.B. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Amer. Math. Soc.*, 43:126–166, 1938.

[14] E. Nakai. Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. *Math. Nachr.*, 166:95–103, 1994.

[15] E. Nakai. The Campanato, Morrey and Holder spaces on spaces of homogeneous type. *Studia Mathematica*, 176:1–19, 2006.

[16] L-E. Persson and N. Samko. Weighted hardy and potential operators in the generalized morrey spaces. *J. Math. Anal. Appl.*, 377:792–806, 2011.

[17] N.G. Samko. Weighted Hardy and potential operators in Morrey spaces. *J. Funct. Spaces and Appl.* to appear.

[18] N.G. Samko. Singular integral operators in weighted spaces with generalized Hölder condition. *Proc. A. Razmadze Math. Inst*, 120:107–134, 1999.

[19] N.G. Samko. On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. *Real Anal. Exch.*, 2004.

[20] N.G. Samko. Weighted Hardy and singular operators in Morrey spaces. *J. Math. Anal. and Appl.*, 350:56–72, 2009.

[21] Y. Sawano, S. Sugano, and H. Tanaka. A note on generalized fractional integral operators on generalized morrey spaces. *Boundary Value Problems*, 2009:doi:10.1155/2009/835865, 2009.

[22] C. T. Zorko. Morrey space. *Proc. Amer. Math. Soc.*, 98(4):586–592, 1986.

[23] D.R. Adams. A note on Choquet integrals with respect to Hausdorff capacity. In *Function spaces and applications (Lund, 1986)*.

[24] D.R. Adams. A note on Riesz potentials. *Duke Math. J.*, 42(4):765–778, 1975.

[25] D.R. Adams and J. Xiao. Nonlinear potential analysis on Morrey spaces and their capacities. *Indiana Univ. Math. J.*, 53(6):1631–1666, 2004.

[26] J. Alvarez. The distribution function in the Morrey space. *Proc. Amer. Math. Soc.*, 83:693–699, 1981.

[27] H. Arai and T. Mizuhara. Morrey spaces on spaces of homogeneous type and estimates for $\Box_b$ and the Cauchy-Szego projection. *Math. Nachr.*, 185(1):5–20, 1997.

[28] V.I. Burenkov, V. Guliev, and H. Guliyev. Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces. *J. Comp. Appl. Math.*, 208(1):280 301, 2007.

[29] V.I. Burenkov and H. Guliyev. Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces. *Studia Math.*, 163(2):157–176, 2004.

[30] F. Chiarenza and M. Frasca. Morrey spaces and Hardy-Littlewood maximal function. *Rend. Math.*, 7:273–279, 1987.

[31] G. Di Fazio and M.A. Ragusa. Commutators and Morrey spaces. *Bollettino U.M.I.*, 7(5-A):323–332, 1991.

[32] Y. Ding and S. Lu. Boundedness of homogeneous fractional integrals on L^p for $n/a < p$. *Nagoya Math. J.*, 167:17–33, 2002.

[33] M. Giaquinta. *Multiple integrals in the calculus of variations and non-linear elliptic systems*. Princeton Univ. Press, 1983.

[34] E.A. Kalita. Dual Morrey spaces. *Dokl. Akad. Nauk*, 361(4):447–449, 1998.

[35] V. Kokilashvili and A. Meskhi. A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces. *Georgian Math. J.*, 16(3):547–551, 2009.

[36] Y. Komori and S. Shirai. Weighted Morrey spaces and a singular integral operator. *Math. Nachr.*, 282(2):219–231, 2009.

[37] A. Kufner, O. John, and S. Fučik. *Function Spaces*. Noordhoff International Publishing, 1977. 454 + XV pages.

[38] C.B. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Amer. Math. Soc.*, 43:126–166, 1938.

[39] E. Nakai. Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. *Math. Nachr.*, 166:95–103, 1994.

[40] E. Nakai. On generalized fractional integrals. *Taiwanese J. Math.*, 5(3):587–602, 2001.

[41] E. Nakai and H. Sumitomo. On generalized Riesz potentials and spaces of some smooth functions. *Sci. Math. Jpn.*, 54(3):463–472, 2001.

[42] D.K. Palagachev and L.G. Softova. Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's. *Potential Analysis*, 20:237–263, 2004.

[43] J. Peetre. On convolution operators leaving $\mathcal{L}^{p,\lambda}$ spaces invariant. *Annali di Mat. Pura ed Appl.*, 72(1):295–304, 1966.

[44] J. Peetre. On the theory of $\mathcal{L}_{p,\lambda}$ spaces. *Function. Analysis*, 4:71–87, 1969.

[45] E.-L. Persson and N. Samko. Weighted Hardy and potential operators in the generalized Morrey spaces. *J. Math. Anal. Appl.*, 377:792–806, 2011.

[46] M.A. Ragusa. Commutators of fractional integral operators on Vanishing-Morrey spaces. *J. of Global Optim.*, 40(1-3):361 – 368, 2008.

[47] N.G. Samko. Weighted Hardy and potential operators in Morrey spaces. *J. Funct. Spaces and Appl.* to appear.

[48] N.G. Samko. Weighted Hardy and singular operators in Morrey spaces. *J. Math. Anal. and Appl.*, 350:56–72, 2009.

[49] Y. Sawano and H. Tanaka. Morrey spaces for non-doubling measures. *Acta Math. Sin. (Engl. Ser.)*, 21(6):1535–1544, 2005.

[50] S. Shirai. Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces. *Hokkaido Math. J.*, 35(3):683–696, 2006.

[51] S. Spanne. Some function spaces defined by using the mean oscillation over cubes. *Ann. Scuola Norm. Sup. Pisa*, 19:593–608, 1965.

[52] G. Stampacchia. The spaces $L^{p,\lambda}, N^{(p,\lambda)}$ and interpolation. *Ann. Scuola Norm. Super. Pisa*, 3(19):443–462, 1965.

[53] M. E. Taylor. *Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, volume 81 of *Math. Surveys and Monogr.* AMS, Providence, R.I., 2000.

[54] C. T. Zorko. Morrey space. *Proc. Amer. Math. Soc.*, 98(4):586–592, 1986.