A sharp estimate of the *k*-modulus of smoothness of Bessel potentials: an application to optimal embeddings

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Tabarz, 19*th* of September 2011

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Outline of the talk

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   - Classical Results
   - Better target spaces
2. Notation and basic definitions
   - Generalized Hölder spaces
   - Bessel-potential spaces
3. Optimal embeddings into generalized Hölder spaces
   - Super-limiting case
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   - General case: characterization of embeddings into $C(\overline{\Omega})$
   - Sharp estimate of the k-modulus of smoothness of Bessel potentials
   - General case and smoothness $\sigma \in (0, n)$
   - Limiting case-Again!
   - Super-limiting case-Again!
4. References

Classical results

Sobolev's classical embedding theorem

$\Omega \subset \mathbb{R}^n$ is a domain with a sufficient smooth boundary

- Limiting case, *i.e.*, $p = \frac{n}{k} \geq 1$

  $$W_p^k(\Omega) \hookrightarrow L_q(\Omega), \quad \text{for all} \quad q \in [p, +\infty),$$

  ($q = +\infty$, if $p = 1$ so that $k = n$).

- Super-limiting case, *i.e.*, $p > n/k$, $W_p^k(\Omega) \hookrightarrow C_B(\Omega)$.

  When $k = 1 + n/p \in \mathbb{N}$,

  $$W_p^k(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \quad \text{for all} \quad \alpha \in (0, 1).$$

  If $p = 1$, we can have $\alpha = 1$ - Lipschitz space.

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Better target spaces

Super-Limiting case: “almost” Lipschitz functions

Brézis-Wainger [1980],

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n})$$

with $\lambda(t) := t|\log t|^{\frac{1}{p'}}$, $t \in (0, \frac{1}{2})$, which implies, for some positive constant c,

$$|f(x) - f(y)| \leq c\|f|H_p^{1+n/p}\|\,|x-y|\,|\log|x-y||^{\frac{1}{p'}}$$

for all $f \in H_p^{1+n/p}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$ such that $0 < |x-y| < \frac{1}{2}$.

Super-Limiting case: “almost” Lipschitz functions

Improvements by

Triebel [2001],

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,p}^{\lambda_p(\cdot)}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda(\cdot)}(\mathbb{R}^n)$$

with $\lambda_p(t) := t|\log t|$, $t \in (0, \frac{1}{2})$, which implies, for some positive constant c, that

$$\|f\|_\infty + \left(\int_0^{1/2} \left(\frac{\omega(f,t)}{t|\log t|} \right)^p \frac{dt}{t} \right)^{1/p} \leq c\|f|H_p^{1+n/p}\|$$

for all $f \in H_p^{1+n/p}(\mathbb{R}^n)$.
$(1 < p < +\infty)$

Example: classical Sobolev space

$n, k \in \mathbb{N}$, $n > 1$, $k \leq n-1$

$f \in W^{k+1,n/k}(\mathbb{R}^n) = W^{k+1}L^{n/k}(\mathbb{R}^n) \quad \Longrightarrow$

(1) $\omega_1(f,t) \lesssim t(1 - log\, t)^{1-k/n}, \quad t \in (0,1)$

Brézis and Wainger, 1980.

(2) $$\int_0^1 \left(\frac{\omega_1(f,t)}{t(1+|\log t|)} \right)^{n/k} \frac{dt}{t} < \infty$$

Better result than (1); consequence of a more general result of Triebel, 2001.

(3) $$\int_0^1 \left(\frac{\omega_2(f,t)}{t} \right)^{n/k} \frac{dt}{t} < \infty$$

Better result than (1) and (2). Result known!!! Also a consequence of more general results of GNO, 2010 ($k < n-1$) and GMNO, 2011 ($k = n-1$).

Example: embedding of Sobolev-Orlicz space

If $k, n \in \mathbb{N}$, $k+1 \le n$ and $\alpha \in \mathbb{R}$,

$$W^{k+1}L^{n/k}(\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow \Lambda^{k+1,\mu(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}).$$

$\mu(t) := t\,\ell^{-\alpha}(t)$ for all $t \in (0,1)$.
Put $p = \frac{n}{k}$ and let $\alpha \le 1 - \frac{k}{n}$. Then,

$$\Lambda^{k+1,\mu(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}) = \Lambda^{2,\mu(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_{pr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_{p\infty}(\cdot)}_{\infty,\infty}(\overline{\mathbb{R}^n}),$$

$$\lambda_{pr}(t) := t\ (\ell(t))^{1-k/n+1/r-\alpha}, \quad t \in (0,1),\ r \in [n/k, +\infty].$$

Taking $\alpha = 0$, we arrive at $W^{k+1,n/k}(\mathbb{R}^n) \hookrightarrow \Lambda^{2,Id(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$
(Brézis-Wainger ($r = +\infty$) and Triebel ($r = p = \frac{n}{k}$) results are a consequence of a better embedding.)

If $\alpha > 1 - \frac{k}{n}$, then, $\Lambda^{k+1,\mu(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,Id(\cdot)}_{\infty,\infty}(\overline{\mathbb{R}^n}) = Lip(\overline{\mathbb{R}^n})$.

Preliminaries: Notation and basic definitions

Notation

$(k-)$ modulus of smoothness:
$h \in \mathbb{R}^n, \quad f \in C_B(\mathbb{R}^n)$

$$\Delta_h f(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n$$

$$\Delta_h^{k+1} f(x) := \Delta_h(\Delta_h^k f)(x), \quad x \in \mathbb{R}^n, \; k \in \mathbb{N}$$

$$\omega_k(f,t) := \sup_{|h| \le t} \|\Delta_h^k f\|_\infty, \quad t \ge 0.$$
$$\omega(f,t) := \omega_1(f,t).$$

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The generalized Hölder space

$k \in \mathbb{N}$, $r \in (0, +\infty]$, $\lambda \in \mathcal{L}_r^k$

$$\Lambda_{\infty,r}^{k;\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \{f \in C_B(\mathbb{R}^n); \, \|f|\Lambda_{\infty,r}^{k,\lambda(\cdot)}(\overline{\mathbb{R}^n})\| < +\infty\}$$

$$\|f|\Lambda_{\infty,r}^{k,\lambda(\cdot)}(\overline{\mathbb{R}^n})\| := \|f\|_\infty + \left\| t^{-1/r} \frac{\omega_k(f,t)}{\lambda(t)} \right\|_{r;(0,1)}$$

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$$\omega(f,t) := \omega_1(f,t).$$

- If $k=1$, we sometimes use $\Lambda_{\infty,r}^{\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \Lambda_{\infty,r}^{1;\lambda(\cdot)}(\overline{\mathbb{R}^n})$ and $C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \Lambda_{\infty,\infty}^{\lambda(\cdot)}(\overline{\mathbb{R}^n})$.
- the scale $\Lambda_{\infty,r}^{k,\lambda(\cdot)}$ contains both Hölder and Zygmund spaces

Bessel-potential spaces $H^\sigma X(\mathbb{R}^n)$

Bessel Kernel

g_σ, com $\sigma > 0$:

$$\widehat{g_\sigma}(\xi) = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\sigma/2}, \xi \in \mathbb{R}^n.$$

where the Fourier transform $\hat{f}$ of a function f is given by

$\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$

Bessel-potential spaces

$X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ r. i. BFS over $(\mathbb{R}^n, \mu_n)$ $\Longrightarrow$

$X \hookrightarrow L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ $\Longrightarrow$

$u = f * g_\sigma$ is well defined for all $f \in X$

$H^\sigma X(\mathbb{R}^n) := \{u : u = f * g_\sigma,\ f \in X(\mathbb{R}^n)\}$

$\|u\|_{H^\sigma X} := \|f\|_X$

Notation: $H^0 X(\mathbb{R}^n) = X(\mathbb{R}^n)$.

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Lorentz-Karamata spaces

Definition

b slowly varying function on $(0,+\infty)$ ($b \in SV(0,+\infty)$):

- Lebesgue measurable function $b : (0,+\infty) \longrightarrow (0,+\infty)$;
- for each $\epsilon > 0$,

$$t^{\epsilon} b(t) \approx f_{\epsilon}(t) \nearrow \text{ and } t^{-\epsilon} b(t) \approx f_{-\epsilon}(t) \searrow .$$

Examples

Let $\alpha, \beta \in \mathbb{R}$:

- $b(t) = (1+|\log t|)^{\alpha}(1+\log(1+|\log t|))^{\beta}$;
- $b(t) = \exp(|\log t|^{\alpha})$, $0 < \alpha < 1$.

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Lorentz-Karamata space $X = L_{p,q;b}(\mathbb{R}^n)$

$\|f\|_{p,q;b} := \|t^{\frac{1}{p}-\frac{1}{q}} b(t)\, f^*(t)\|_{q;(0,+\infty)}$

(f^* *non-increasing* rearrangement of $f \in \mathcal{M}(\mathbb{R}^n, \mu_n)$).

Example

$b = \ell^{\boldsymbol{\alpha}}$ - product of powers of iterated “logs”

Logarithmic Bessel-potential-type spaces (Edmunds, Gurka and Opic, 1997).

Optimal Embeddings: super-limiting case

Theorem [Gogatishvili, N. & Opic (2005)]

Let $\sigma \in [1, n+1)$, $\max\{1, n/\sigma\} < p < n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ and let $b \in SV(0, +\infty)$. $\Omega \subset \mathbb{R}^n$ a nonempty domain. $\lambda(t) = t^{\sigma - n/p}[b(t^n)]^{-1}, \quad t \in (0, 1]$.

(i) Then $H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$.

(ii) If $\lim_{t \to 0_+} \frac{\frac{t}{\lambda(t)}}{\|\tau^{-1/r} \frac{\tau}{\mu(\tau)}\|_{r;(0,t)}} = 0$, $H^\sigma L_{p,q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$

(iii) Let $\overline{q} \in (0, q)$. $H^\sigma L_{p,q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,\overline{q}}(\overline{\Omega})$.

(i) - N. (2004); (i),(ii)- Logarithmic Bessel potential space-Edmunds, Gurka and Opic (1997,2000) - $r = +\infty$;

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Let $\sigma \in (1, n+1)$, $p = n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ and $b \in SV(0, +\infty)$ such that $\| t^{-1/q'} [b(t)]^{-1} \|_{q';(0,1)} = +\infty$.

$$\lambda_r(t) = t\,[b(t^n)]^{q'/r} \left(\int_{t^n}^{2} \tau^{-1} [b(\tau)]^{-q'} d\tau \right)^{1/q'+1/r}, \quad t \in (0,1].$$

(i) Then $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\mathbb{R}^n})$.

(ii) If $\lim_{t\to 0_+} \dfrac{\|\tau^{-1/r} \frac{\tau}{\lambda_r(\tau)}\|_{r;(0,t)}}{\|\tau^{-1/r} \frac{\tau}{\mu(\tau)}\|_{r;(0,t)}} = 0$, $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\mu(.)}_{\infty,r}(\overline{\Omega})$

(iii) Let $\overline{q} \in (0, q)$. $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\lambda_{\overline{q}}(.)}_{\infty,\overline{q}}(\overline{\Omega})$.

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Optimal Embeddings: super-limiting case

Proof: [(i)] general ideas

- $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.
- Let $u \in \mathcal{S}(\mathbb{R}^n) \subset H^\sigma L_{p,q;b}(\mathbb{R}^n)$.
- (Lifting argument) $\frac{\partial u}{\partial x_i} \in H^{\sigma-1} L_{p,q;b}(\mathbb{R}^n)$, for $i = 1, \ldots, n$.
- Use embedding results for limiting case and inequality of DeVore and Sharpley.
- Use of appropriate Hardy-type inequalities.
- Hence $\|u|\Lambda^{\lambda(\cdot)}_{\infty,r}\| \precsim \|u\|_{\sigma;p,q;b}$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Proofs: [(ii), (iii)] general ideas

- Use of appropriate extremal functions!

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Theorem

Let $k \in \mathbb{N}$, $\sigma \in [k, +\infty)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be an r.i. BFS. Suppose that the Boyd indices of X belong to the interval $(0,1)$ and that the space X has absolutely continuous norm. Then

$$H^\sigma X(\mathbb{R}^n) = W^k H^{\sigma-k} X(\mathbb{R}^n).$$

In particular,

$$H^k X(\mathbb{R}^n) = W^k X(\mathbb{R}^n).$$

Example

When $k \in \mathbb{N}$, $p \in (1, +\infty)$, $q \in [1, +\infty)$ and $b \in SV(0, +\infty)$, then

$$H^k L_{p,q;b}(\mathbb{R}^n) = W^k L_{p,q;b}(\mathbb{R}^n).$$

- Use embedding results for limiting case and inequality of DeVore and Sharpley.
- Use of appropriate Hardy-type inequalities.

Optimal Embeddings: super-limiting case

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- Use embedding results for limiting case and inequality of DeVore and Sharpley.

Theorem (DeVore and Sharpley, 1984)

A function f such that $|\nabla f| \in L^{n,1}_{loc}(\mathbb{R}^n)$ can be redefined on a set of measure zero so that f is continuous on $\mathbb{R}^n$ and the modulus of smoothness $\omega_1(f,\cdot)$ satisfies the inequality

$$\omega_1(f,t) \precsim \int_0^{t^n} s^{\frac{1}{n}-1} |\nabla f|^*(s)\, ds \quad \textit{for all} \quad t \in (0,1).$$

- Use of appropriate Hardy-type inequalities.
- Hence $\|u|\Lambda^{\lambda(\cdot)}_{\infty,r}\| \precsim \|u\|_{\sigma;p,q;b}$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

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Proofs: [(ii), (iii)] general ideas

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Optimal Embeddings: limiting case

And when we cannot use Lifting argument?
The bellow example is proved by different methods and was the motivation to look for different approaches!

Theorem (Gogatishvili, N. and Opic (2007))

Let $0 < \sigma < n$, $p = n/\sigma$, $q \in (1, +\infty)$ *and* $b \in SV(0, +\infty)$ *such that* $\|t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$. *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}).$$

where

$$\lambda(t) := \left(\int_0^{t^n} [b(\tau)]^{-q'} \, \frac{dt}{t} \right)^{1/q'}, \quad t > 0.$$

Logarithmic Bessel potential space - Edmunds, Gurka and Opic (2005);

$$H^\sigma L_{p,q;\frac{1}{q'},\cdots,\frac{1}{q'},\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}),$$

with $\alpha_m > \frac{1}{q'}$ and $\lambda(t) = \ell_m^{\frac{1}{q'}-\alpha_m}(t)$.

General case: characterization of embeddings into $C(\overline{\Omega})$

Theorem [Gogatishvili, N. & Opic (2009) $\sigma < 1$, (2010) $\sigma \in (0, n)$]

Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be a r. i. BFS. Assume that Ω is a domain in $\mathbb{R}^n$. Then $H^\sigma X(\mathbb{R}^n) \hookrightarrow C(\overline{\Omega})$ if and only if $\|g_\sigma\|_{X'} < +\infty$.

Lemma [Gogatishvili, N. & Opic (2009) $\sigma < 1$, (2010) $\sigma \in (0, n)$]

Let $\sigma \in (0, n)$, $p \in (1, +\infty)$, $q \in [1, +\infty]$ and $b \in SV(0, +\infty)$. If $X = L_{p,q;b}(\mathbb{R}^n)$, then

$$g_\sigma \in X'$$

if and only if either

$$p > \frac{n}{\sigma} \tag{1}$$

or

$$p = \frac{n}{\sigma} \quad \text{and} \quad \|t^{-\frac{1}{q'}}(b(t))^{-1}\|_{q';(0,1)} < +\infty. \tag{2}$$

Key estimates of the k-modulus of smoothness of Bessel potentials

Theorem

Let $\sigma \in (0, n)$ *and let* $X = X(\mathbb{R}^n)$ *be an r. i. BFS such that* $\|g_\sigma\|_{X'} < +\infty$*.*
Then $f * g_\sigma \in C(\overline{\mathbb{R}^n})$ *for all* $f \in X$ *and*

$$\omega_k(f * g_\sigma, t) \precsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \; \forall t \in (0,1), \forall f \in X,$$

where $k \geq [\sigma] + 1$*.*
Moreover, given $k \in \mathbb{N}$*, there are* $\delta \in (0,1)$ *and* $\alpha > 0$ *such that*

$$\omega_k(\bar{f} * g_\sigma, t) \succsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \; \forall t \in (0,1), \forall f \in X,$$

where $\bar{f}(x) := f^*(\beta_n |x|^n) \chi_{C_\alpha(0,\delta)}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$*,*
$C_\alpha(0,\delta) := C_\alpha \cap B(0,\delta)$ *with* $C_\alpha := \{y \in \mathbb{R}^n : \; y_1 > 0, \; y_1^2 > \alpha \sum_{i=2}^n y_i^2\}$*.*

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Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be an r. i. BFS such that $\|g_\sigma\|_{X'} < +\infty$. Then $f * g_\sigma \in C(\overline{\mathbb{R}^n})$ for all $f \in X$ and

$$\omega_k(f * g_\sigma, t) \precsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \; \forall t \in (0,1), \forall f \in X,$$

where $k \geq [\sigma] + 1$.
Moreover, given $k \in \mathbb{N}$, there are $\delta \in (0,1)$ and $\alpha > 0$ such that

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where $\bar{f}(x) := f^*(\beta_n |x|^n) \chi_{C_\alpha(0,\delta)}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,
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Moreover, given $k \in \mathbb{N}$, *there are* $\delta \in (0,1)$ *and* $\alpha > 0$ *such that*

$$\omega_k(\bar{f} * g_\sigma, t) \succsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \; \forall t \in (0,1), \forall f \in X,$$

where $\bar{f}(x) := f^*(\beta_n|x|^n)\chi_{C_\alpha(0,\delta)}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,
$C_\alpha(0,\delta) := C_\alpha \cap B(0,\delta)$ *with* $C_\alpha := \{y \in \mathbb{R}^n : \; y_1 > 0, \; y_1^2 > \alpha \sum_{i=2}^n y_i^2\}$.

One of the key steps in the proof of the sharpness of this estimate is the assertion that sgn $\frac{\partial^j g_\sigma}{\partial x_1^j}(x) = (-1)^j$, $\sigma \in (0,n)$, $j \in \mathbb{N}$, for all x in a small circular half-cone whose vertex is at the origin and whose axis coincides with the positive part of x_1-axis.

Optimal Embeddings: general case and smoothness $\sigma \in (0, n)$

Theorem [Gogatishvili, N. & Opic (2010)]

Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be a r. i. BFS such that $\|g_\sigma\|_{X'} < +\infty$. Put $k := [\sigma] + 1$, assume that $r \in (0, +\infty]$ and $\mu \in \mathcal{L}_r^k$. Then

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\mathbb{R}^n})$$

if and only if

$$\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\, d\tau \right\|_{r;(0,1)} \precsim \|f\|_X \quad \text{for all} \quad f \in X.$$

Optimal Embeddings: limiting case-Example

Theorem [Gogatishvili, N. & Opic (2009) $\sigma < 1$, [Gogatishvili, N. & Opic (2010) $\sigma \in (0, n)$]

Let $\sigma \in (0, n)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$, $r \in (0, +\infty]$, $k = [\sigma] + 1$, $\mu \in \mathcal{L}_r^k$ and let $b \in SV(0, +\infty)$: $\|t^{-\frac{1}{q'}}(b(t))^{-1}\|_{q';(0,1)} < +\infty$. Let

$$\lambda_{qr}(x) := b^{q'/r}(x^n)\left(\int_0^{x^n} b^{-q'}(t)\,\frac{dt}{t}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad x \in (0, 1].$$

If $1 < q \leq r \leq +\infty$, then

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\mathbb{R}^n}) \iff \overline{\lim_{x\to 0_+}} \frac{\|t^{-\frac{1}{r}}(\mu(t))^{-1}\|_{r;(x,1)}}{\|t^{-\frac{1}{r}}(\lambda_{qr}(t))^{-1}\|_{r;(x,1)}} < +\infty.$$

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Remark

When $r \in [1, +\infty]$, then

$$\Lambda_{\infty,r}^{k,\lambda_{qr}(\cdot)}(\overline{\mathbb{R}^n}) = \Lambda_{\infty,r}^{1,\lambda_{qr}(\cdot)}(\overline{\mathbb{R}^n}).$$

Embeddings: Examples in the limiting case

Example

Let $\sigma \in (0, n)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$ and $r \in [q, +\infty]$.
If $\alpha > \frac{1}{q'}$, $\beta \in \mathbb{R}$, then

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

$$\lambda_{qr}(t) = (1 + |\log t|)^{\frac{1}{r}+\frac{1}{q'}-\alpha}(1 + \log(1 + |\log t|))^{-\beta}, \quad t \in (0, 1];$$
$$\lambda_{q\infty}(t) = (1 + |\log t|)^{\frac{1}{q'}-\alpha}(1 + \log(1 + |\log t|))^{-\beta}, \quad t \in (0, 1].$$

If $\alpha = \frac{1}{q'}$, $\beta > \frac{1}{q'}$, then

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

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Embeddings: Examples in the limiting case

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Optimal Embeddings: super-limiting case-Again!

Theorem [Gogatishvili, N. & Opic (2009) $\sigma < 1$, [Gogatishvili, N. & Opic (2010) $\sigma \in (0, n)$]

Let $\sigma \in (0, n)$, $p \in (\frac{n}{\sigma}, +\infty)$, $q \in [1, +\infty]$, $b \in SV(0, +\infty)$, $r \in (0, +\infty]$, $k = [\sigma] + 1$, and $\mu \in \mathcal{L}_r^k$.
Let

$$\lambda(x) := x^{\sigma - \frac{n}{p}}(b(x^n))^{-1}, \quad x \in (0, 1].$$

If $1 \leq q \leq r \leq +\infty$, then

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\mathbb{R}^n}) \iff \overline{\lim_{x \to 0_+}} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,1)} \lambda(x) < +\infty.$$

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Let

$$\lambda(x) := x^{\sigma - \frac{n}{p}} (b(x^n))^{-1}, \quad x \in (0, 1].$$

If $1 \leq q \leq r \leq +\infty$, then

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \Longleftrightarrow \overline{\lim_{x \to 0_+}} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,1)} \lambda(x) < +\infty.$$

Remark

When $r \in [1, +\infty]$, then

$$\Lambda^{k,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{[\sigma - \frac{n}{p}]+1,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}).$$

If $\sigma = \frac{n}{p} + 1$, then

$$\Lambda^{k,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{2,\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}).$$

Optimal Embeddings: super-limiting case-Again!

Corollary [Gogatishvili, N. & Opic (2010)]

If $\sigma \in (1, n)$, $p = \frac{n}{\sigma - 1}$, $q \in (1, +\infty]$, $r \in [q, +\infty]$ and $b \in SV(0, +\infty)$ be such that $\| t^{-1/q'} [b(t)]^{-1} \|_{q';(0,1)} = +\infty$.
Let

$$\lambda(t) := t\,(b(t^n))^{-1}, \quad t \in (0, 1].$$

and let

$$\lambda_{qr}(t) := t\,[b(t^n)]^{q'/r} \left(\int_{t^n}^{2} \tau^{-1} [b(\tau)]^{-q'} \, d\tau \right)^{1/q' + 1/r}, \quad t \in (0, 1].$$

Then

$$H^{\sigma} L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{2,\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}).$$

Remark

Previous Corollary improves Theorem 3.2 of (GNO, 2005), provided $\sigma = 1 + \frac{n}{p} < n$ and shows that the Brézis-Wainger embedding of the Sobolev space $H_p^{1+\frac{n}{p}}(\mathbb{R}^n)$, $\sigma = 1 + \frac{n}{p} < n$, into the space of “almost” Lipschitz functions is a consequence of a better embedding whose target is the Zygmund space.

Example

if $n > 1$, $p = q$, $p > \frac{n}{n-1}$, $b(t) = \ell_1^{\alpha}(t)$, $t \in (0, +\infty)$, $\alpha < \frac{1}{p'}$,

$$H^{1+\frac{n}{p}} L^p(\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow \Lambda^{2,\lambda(\cdot)}_{\infty,p}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_r(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_\infty(\cdot)}_{\infty,\infty}(\overline{\mathbb{R}^n}),$$

with

$$\lambda(x) := x\,\ell_1^{-\alpha}(x) \quad \text{for all} \quad x \in (0,1],$$

and

$$\lambda_{pr}(x) = x\ (\ell_1(x))^{1/p'+1/r-\alpha}, \quad x \in (0,1],\ r \in [p, +\infty].$$

If $\alpha = 0$, this example shows that the Brézis-Wainger embedding of the Sobolev space $H_p^{1+\frac{n}{p}}(\mathbb{R}^n)$, $\sigma = 1 + \frac{n}{p} < n$, into the space of "almost" Lipschitz functions is a consequence of a better embedding whose target is the Zygmund space $\Lambda^{2,Id(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$ ($Id(\cdot)$ stands for the identity map).

Example

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