B^σ-function spaces and integral operators

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This is a joint work with

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1 Introduction

Beurling (1964) introduced the space $B^p(\mathbb{R}^n)$ together with its predual $A^p(\mathbb{R}^n)$, so-called the Beurling algebra. Later, so as to extend Wiener's ideas (1930, 1932), which describe the behavior of a function at infinity, Feichtinger (1984) gave an equivalent norm on $B^p(\mathbb{R}^n)$, which is a special case of norms to describe non-homogeneous Herz spaces $K^{\alpha}_{p,r}(\mathbb{R}^n)$ (1968).

The function space $B^p(\mathbb{R}^n)$ and its homogeneous version $\dot{B}^p(\mathbb{R}^n)$ are characterized by the following norms, respectively:

$$\|f\|_{B^p} = \sup_{r\geq 1} \frac{1}{r^{n/p}} \|f\|_{L^p(Q_r)} \quad \text{and} \quad \|f\|_{\dot{B}^p} = \sup_{r>0} \frac{1}{r^{n/p}} \|f\|_{L^p(Q_r)}. \tag{1.1}$$

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$$\|f\|_{L^p(Q_r)} \Rightarrow \|f\|_{E(Q_r)}, \quad \frac{1}{r^{n/p}} \Rightarrow \frac{1}{r^{\sigma}}.$$

We denote these function spaces by $B^{\sigma}(E)(\mathbb{R}^n)$ and $\dot{B}^{\sigma}(E)(\mathbb{R}^n)$, respectively.

Example 1.1. If $E = L^p/\mathcal{C}$ with $\|f\|_{E(Q_r)} = \|f - f_{Q_r}\|_{L^p(Q_r)}$ and $\sigma = n/p$, then $B^\sigma(E)(\mathbb{R}^n)$ and $\dot{B}^\sigma(E)(\mathbb{R}^n)$ coincide with the central mean oscillation space $\mathrm{CMO}^p(\mathbb{R}^n)$ and the central bounded mean oscillation space $\mathrm{CBMO}^p(\mathbb{R}^n)$, respectively, where

$$f_{Q_r} = \frac{1}{|Q_r|}\int_{Q_r} f(y)\,dy, \quad |Q_r| \text{ is the Lebesgue measure of } Q_r.$$

In this case

$$\|f\|_{\mathrm{CMO}^p} = \sup_{r\geq 1}\frac{1}{r^{n/p}}\|f - f_{Q_r}\|_{L^p(Q_r)},$$
$$\|f\|_{\mathrm{CBMO}^p} = \sup_{r>0}\frac{1}{r^{n/p}}\|f - f_{Q_r}\|_{L^p(Q_r)}.$$

$\mathbf{CMO}^p$ is introduced by Chen and Lau (1989) and García-Cuerva (1989).

$\mathbf{CBMO}^p$ is introduced by Lu and Yang (1992,1995).

Example 1.2. $\boldsymbol{B}^{p,\lambda}(\mathbb{R}^n), \dot{\boldsymbol{B}}^{p,\lambda}(\mathbb{R}^n), \mathbf{CMO}^{p,\lambda}(\mathbb{R}^n), \mathbf{CBMO}^{p,\lambda}(\mathbb{R}^n)$:

$$\|f\|_{B^{p,\lambda}} = \sup_{r\geq 1} \frac{1}{r^{n/p+\lambda}} \|f\|_{L^p(Q_r)},$$

$$\|f\|_{\dot{B}^{p,\lambda}} = \sup_{r>0} \frac{1}{r^{n/p+\lambda}} \|f\|_{L^p(Q_r)},$$

$$\|f\|_{\mathrm{CMO}^{p,\lambda}} = \sup_{r\geq 1} \frac{1}{r^{n/p+\lambda}} \|f - f_{Q_r}\|_{L^p(Q_r)},$$

$$\|f\|_{\mathrm{CBMO}^{p,\lambda}} = \sup_{r>0} \frac{1}{r^{n/p+\lambda}} \|f - f_{Q_r}\|_{L^p(Q_r)}.$$

These function spaces are introduced by García-Cuerva and Herrero (1994) and Alvarez, Guzmán-Partida and Lakey (2000) as an extension of $\boldsymbol{B}^p(\mathbb{R}^n)$, $\dot{\boldsymbol{B}}^p(\mathbb{R}^n)$, $\mathbf{CMO}^p(\mathbb{R}^n)$ and $\mathbf{CBMO}^p(\mathbb{R}^n)$.

We also consider the weak L^p spaces WL^p with the semi norm

$$\|f\|_{WL^p(U)} = \sup_{t>0} t\, m(U,f,t)^{1/p}, \quad m(U,f,t) = |\{y \in U : |f(y)| > t\}|,$$

where $U \subset \mathbb{R}^n$. We shall consider the case $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$.

EXAMPLE **1.3.** If $E = WL^p$ and $\sigma = n/p + \lambda$, $\lambda \in \mathbb{R}$, then $B^\sigma(E)(\mathbb{R}^n)$ and $\dot{B}^\sigma(E)(\mathbb{R}^n)$ coincide with the weak central Morrey spaces $WB^{p,\lambda}(\mathbb{R}^n)$ and $W\dot{B}^{p,\lambda}(\mathbb{R}^n)$, respectively. In this case

$$\|f\|_{WB^{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^{n/p+\lambda}} \|f\|_{WL^p(Q_r)}, \quad \|f\|_{W\dot{B}^{p,\lambda}} = \sup_{r>0} \frac{1}{r^{n/p+\lambda}} \|f\|_{WL^p(Q_r)}. \quad (1.2)$$

2 Notation and definitions

Let

$$Q_r = \left\{ y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \max_{1 \le i \le n} |y_i| < r \right\} \text{ or } Q_r = \{ y \in \mathbb{R}^n : |y| < r \},$$

and let

$$Q(x,r) = x + Q_r = \{ x + y : y \in Q_r \}.$$

For $Q = Q(x,r)$, let

$$f_Q = \fint_Q f(y)\,dy = \frac{1}{|Q|} \int_Q f(y)\,dy \tag{2.1}$$

and

$$m(Q,f,t) = |\{ y \in Q : |f(y)| > t \}| \quad \text{and} \quad \underline{m}(Q,f,t) = \frac{m(Q,f,t)}{|Q|}. \tag{2.2}$$

We recall the definition of Morrey, weak Morrey, Campanato and Lipschitz spaces.

Definition 2.1. Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $p \in [1, \infty)$, $\lambda \in \mathbb{R}$ and $\alpha \in (0, 1]$, let $L_{p,\lambda}(U)$, $WL_{p,\lambda}(U)$, $\mathcal{L}_{p,\lambda}(U)$, $\mathrm{Lip}_\alpha(U)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\|f\|_{L_{p,\lambda}(U)} = \sup_{Q(x,s)\subset U} \frac{1}{s^\lambda} \left(\fint_{Q(x,s)} |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{WL_{p,\lambda}(U)} = \sup_{Q(x,s)\subset U} \frac{1}{s^\lambda} \sup_{t>0} t \, \underline{m}(Q(x,s), f, t)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}(U)} = \sup_{Q(x,s)\subset U} \frac{1}{s^\lambda} \left(\fint_{Q(x,s)} |f(y) - f_{Q(x,s)}|^p \, dy \right)^{1/p},$$

and

$$\|f\|_{\mathrm{Lip}_\alpha(U)} = \sup_{x,y\in U,\, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Definition 2.2. Let $\sigma \in [0,\infty)$, $p \in [1,\infty)$, $\lambda \in \mathbb{R}$ and $\alpha \in (0,1]$. For

$$E = L^p,\ WL^p,\ L_{p,\lambda},\ WL_{p,\lambda},\ \mathcal{L}_{p,\lambda},\ \mathrm{BMO} \text{ or } \mathrm{Lip}_\alpha,$$

let

$$B^\sigma(E)(\mathbb{R}^n) = \left\{ f : \|f\|_{B^\sigma(E)} = \sup_{r\ge 1} \frac{1}{r^\sigma} \|f\|_{E(Q_r)} < \infty \right\},$$
$$\dot{B}^\sigma(E)(\mathbb{R}^n) = \left\{ f : \|f\|_{\dot{B}^\sigma(E)} = \sup_{r>0} \frac{1}{r^\sigma} \|f\|_{E(Q_r)}. < \infty \right\}.$$

Remark 2.1. We note that $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ unifies $L_{p,\lambda}(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ and that $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unifies $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$. Actually, we have the following relations:

$$B^0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n), \quad B^0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n), \tag{2.3}$$

$$B^{\lambda+n/p}(L_{p,-n/p})(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n), \quad B^{\lambda+n/p}(\mathcal{L}_{p,-n/p})(\mathbb{R}^n) = \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n). \tag{2.4}$$

The function spaces $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ also have the same properties.

3 Sublinear operators

First we consider the fractional maximal operators M_α of order $\alpha \in [0, n)$, which is defined as

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy, \tag{3.1}$$

where the supremum is taken over all cubes (or balls) Q containing $x \in \mathbb{R}^n$. If $\alpha = 0$, then M_α is the Hardy-Littlewood maximal operator denoted by M.

It is known that,

$$M_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad \text{if } p \in (1, \infty],$$
$$M_\alpha : L^1(\mathbb{R}^n) \to WL^q(\mathbb{R}^n) \quad \text{if } p = 1,$$

for $\alpha \in [0, n)$, $p, q \in [1, \infty]$ and $-n/p + \alpha = -n/q$.

Our first main result is the following:

THEOREM **3.1.** *Let* $\boldsymbol{\alpha} \in [\mathbf{0}, \boldsymbol{n})$, $\boldsymbol{\sigma} \in [\mathbf{0}, \infty)$ *and* $\boldsymbol{p}, \boldsymbol{q} \in [\mathbf{1}, \infty)$, *and let* $\boldsymbol{\lambda} \in [-\boldsymbol{n}/\boldsymbol{p}, \mathbf{0})$ *and* $\boldsymbol{\mu} \in [-\boldsymbol{n}/\boldsymbol{q}, \mathbf{0})$. *Assume that*

$$\boldsymbol{\mu} = \boldsymbol{\lambda} + \boldsymbol{\alpha}, \quad \boldsymbol{q} \le (\boldsymbol{\lambda}/\boldsymbol{\mu})\boldsymbol{p} \quad \text{and} \quad \boldsymbol{\sigma} + \boldsymbol{\lambda} + \boldsymbol{\alpha} \le \mathbf{0}.$$

Then

$$\boldsymbol{M_\alpha} : \boldsymbol{B^\sigma}(\boldsymbol{L_{p,\lambda}})(\mathbb{R}^{\boldsymbol{n}}) \to \boldsymbol{B^\sigma}(\boldsymbol{L_{q,\mu}})(\mathbb{R}^{\boldsymbol{n}}), \quad \text{if } \boldsymbol{p} \in (\mathbf{1}, \infty),$$

$$\boldsymbol{M_\alpha} : \boldsymbol{B^\sigma}(\boldsymbol{L_{1,\lambda}})(\mathbb{R}^{\boldsymbol{n}}) \to \boldsymbol{B^\sigma}(\boldsymbol{W L_{q,\mu}})(\mathbb{R}^{\boldsymbol{n}}), \quad \text{if } \boldsymbol{p} = \mathbf{1}.$$

The same conclusion holds for $\dot{\boldsymbol{B}}^{\boldsymbol{\sigma}}(\boldsymbol{L_{p,\lambda}})(\mathbb{R}^{\boldsymbol{n}})$.

REMARK 3.1. Let $\boldsymbol{\alpha} = \mathbf{0}$ in the theorem above. Then we have the boundedness of the Hardy-Littlewood maximal operator.

The theorem contains the results of

Chiarenza and Frasca (1987) ($\boldsymbol{M}$ on $\boldsymbol{L_{p,\lambda}}(\mathbb{R}^{\boldsymbol{n}})$),

García-Cuerva (1989) ($\boldsymbol{M}$ on $\boldsymbol{B^p}(\mathbb{R}^{\boldsymbol{n}})$).

Next, we consider more general sublinear operators T which satisfies the following condition: For some $\alpha \in [0, n)$,

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy, \quad x \notin \operatorname{supp} f, \quad f \in L^1_{\mathrm{comp}}(\mathbb{R}^n), \tag{3.2}$$

where Ω is homogeneous of degree zero and $\Omega \in L^{\tilde{p}}(S^{n-1})$ for some $\tilde{p} \in [1, \infty]$.

For example, Carleson's maximal operator, C. Fefferman's singular multipliers, Ricci-Stein's oscillatory singular integral, the Bochner-Riesz operator at the critical index, R. Fefferman's singular integral, and so on; In particular, the Calderón-Zygmund singular integral operator, and fractional integral operators satisfy (3.2) with $\Omega \equiv 1$. More precisely, the Calderón-Zygmund singular integral operator T is defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy, \quad x \notin \operatorname{supp} f, \quad f \in L^1_{\mathrm{comp}}(\mathbb{R}^n) \tag{3.3}$$

with kernel $K(x, y)$ satisfying the condition

$$|K(x, y)| \le C|x-y|^{-n}, \quad x \neq y, \tag{3.4}$$

and some regularity conditions.

Fractional integral operators I_α, $0 < \alpha < n$, are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy. \tag{3.5}$$

Then I_α satisfies (3.2) with this α and it is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p < q < \infty$, $-n/p + \alpha = -n/q$, and from $L^1(\mathbb{R})$ to $WL^{n/(n-\alpha)}(\mathbb{R})$.

Our second result is the following:

Theorem 3.2. *Let* $\sigma \in [0,\infty)$ *and* $p, q \in [1,\infty)$, *and let* $\lambda \in [-n/p, 0)$ *and* $\mu \in [-n/q, 0)$. *Let* T *be a sublinear operator defined on* $L^1_{\mathrm{comp}}(\mathbb{R}^n)$ *and satisfy* (3.2) *for some* $\alpha \in [0, n)$ *and* $\Omega \in L^{\tilde{p}}(S^{n-1})$ *with* $\tilde{p} \geq p'$. *Assume that*

$$\mu = \lambda + \alpha \quad \textit{and} \quad \sigma + \lambda + \alpha < 0.$$

Assume in addition that $T : L_{p,\lambda}(\mathbb{R}^n) \to L_{q,\mu}(\mathbb{R}^n)$ *or* $T : L_{p,\lambda}(\mathbb{R}^n) \to WL_{q,\mu}(\mathbb{R}^n)$, *Then*

$$T : B^{\sigma}(L_{p,\lambda})(\mathbb{R}^n) \to B^{\sigma}(L_{q,\mu})(\mathbb{R}^n) \quad \textit{or} \quad T : B^{\sigma}(L_{p,\lambda})(\mathbb{R}^n) \to B^{\sigma}(WL_{q,\mu})(\mathbb{R}^n),$$

respectively. The same conclusion holds for $\dot{B}^{\sigma}(L_{p,\lambda})(\mathbb{R}^n)$.

The theorem contains the results of

- Adams (1975) (I_α on $L_{p,\lambda}$),
- Chiarenza and Frasca (1987), N (1994) (T with $\Omega \equiv 1$ on $L_{p,\lambda}(\mathbb{R}^n)$),
- Ding, Yang and Zhow (1998) (T on $L_{p,\lambda}(\mathbb{R}^n)$),
- Fu, Lin and Lu (2008) (T on $B^{p,\lambda}$)

4 Singular integral operators with the cancellation property

Let $\kappa \in (0,1]$. In this section we consider a singular integral operator T with kernel $K(x,y)$ satisfying the following properties;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for } x \neq y; \tag{4.1}$$

$$|K(x,y)-K(z,y)| + |K(y,x)-K(y,z)| \le \frac{C}{|x-y|^n}\left(\frac{|x-z|}{|x-y|}\right)^{\kappa} \quad \text{for } |x-y| \ge 2|x-z|; \tag{4.2}$$

$$\int_{r\le|x-y|<R} K(x,y)\,dy = \int_{r\le|x-y|<R} K(y,x)\,dy = 0 \quad \text{for } 0<r<R<\infty \text{ and } x\in\mathbb{R}^n, \tag{4.3}$$

where C is a positive constant independent of $x,y,z \in \mathbb{R}^n$.

To define T for B^σ-Campanato spaces, we first define the modified version of T as follows:

$$\tilde{T}f(x) = \int_{\mathbb{R}^n} \big[K(x,y) - K(0,y)(1-\chi_1(y))\big] f(y)\,dy. \tag{4.4}$$

Our result is the following.

THEOREM 4.1. *Let* T *be a singular integral operator of type* $\kappa \in (0,1]$. *Let* $\sigma \in [0,\infty)$ *and* $p \in (1,\infty)$. *If* $-n/p+\sigma<\kappa$ *and if* $\lambda \in [-n/p, \kappa-\sigma)$, *then*

$$\tilde{T} : B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C} \to B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}, \quad \tilde{T} : B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) \to B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n).$$

Moreover, if $\sigma<\kappa$ *and if* $\lambda \in [0,\kappa-\sigma)$, *then*

$$\tilde{T} : B^\sigma(\mathcal{L}_{1,\lambda})(\mathbb{R}^n)/\mathcal{C} \to B^\sigma(\mathcal{L}_{1,\lambda})(\mathbb{R}^n)/\mathcal{C}, \quad \tilde{T} : B^\sigma(\mathcal{L}_{1,\lambda})(\mathbb{R}^n) \to B^\sigma(\mathcal{L}_{1,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ *and* $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$.

Note that

$$(B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C},\ \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}) \quad \text{and} \quad (B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n),\ \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}+|f_{Q_1}|)$$

are Banach spaces.

The theorem contains the results of

Peetre (1966), N (2010) (T on $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$).

5 Fractional integral operators on B^σ-Morrey-Campanato spaces

We define the modified version of I_α as follows;

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_1(y)}{|y|^{n-\alpha}} \right) dy.$$

THEOREM 5.1. *Let* $0 < \alpha < n$, $1 \le p < \infty$, $-n/p + \alpha \le \lambda + \alpha = \mu < 1$ *and* $0 \le \sigma < -\lambda - \alpha + 1$. *Assume that* p *and* q *satisfy one of the following conditions:*

(i) $p = 1$ *and* $1 \le q < n/(n-\alpha)$;

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n-p\alpha)$;

(iii) $n/\alpha \le p < \infty$ *and* $1 \le q < \infty$ (*in this case,* $0 \le \mu < 1$).

Then $\tilde{I}_\alpha : B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \to B^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$, *that is,*

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathcal{L}_{q,\mu})} + |(\tilde{I}_\alpha f)_{B_1}| \le C\|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ *to* $\dot{B}^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$.

6 Other results

Moreover, on B^σ-function spaces, we have

wighted version,

vector valued version,

multilinear version,

commutator,

Littlewood-Paley characterization,

and so on........

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