Sharp conditions on multilinear Fourier multipliers

FSDONA 2011, Tabarz

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Bilinear Fourier multiplier operator

For $m \in L^\infty(\mathbb{R}^{2n})$, we define

$$T_m(f_1, f_2)(x) \\ = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix(\xi_1+\xi_2)} m(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2)\, d\xi_1 d\xi_2,$$

where f_1, $f_2 \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The function m is called the multiplier. With $K = \mathcal{F}^{-1}m$, we have

$$T_m(f_1, f_2)(x) \\ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y_1, x - y_2) f_1(y_1) f_2(y_2)\, dy_1 dy_2.$$

We shall consider m that satisfy, in certain weak sense,

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2)| \le C_{\alpha_1,\alpha_2}(|\xi_1|+|\xi_2|)^{-|\alpha_1|-|\alpha_2|}$$

and we want to show the boundedness

$$T_m : H^{p_1} \times H^{p_2} \longrightarrow L^p, \qquad 1/p_1 + 1/p_2 = 1/p,$$

where H^{p_1} and H^{p_2} are the usual Hardy spaces on $\mathbb{R}^n$. We adopt the convention that $H^{p_i} = L^{p_i}$ for $1 < p_i \le \infty$.

In the case $p_1 = p_2 = p = \infty$, we replace $H^{p_1} \times H^{p_2} \longrightarrow L^p$ by $L^\infty \times L^\infty \to BMO$.

The basic results are the following.

Coifman-Meyer (1978, 1984, 1997): $p_1, p_2, p > 1$.

Kenig-Stein (1999): $p_1, p_2 > 1$, $p > 1/2$.

Grafakos-Kalton (2001): $p_1, p_2, p > 0$.

To assure the boundedness of T_m, it is sufficient to assume the condition

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)| \le C_{\alpha_1,\alpha_2}(|\xi_1| + |\xi_2|)^{-|\alpha_1|-|\alpha_2|} \tag{1}$$

for derivatives up to certain order. In the papers cited above, the authors are mostly assuming (1) for $|\alpha_1| + |\alpha_2| \le 2n + 1$.

In this talk we shall consider the problem to find "sharp" differentiability conditions of the type (1) that assure the boundedness of T_m.

Sharp conditions

Recall the case of linear Fourier multiplier operator:

$$m(D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi) \widehat{f}(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n).$$

It is well-known that $m(D) : H^p \to H^p$, $0 < p < \infty$, if $m(\xi)$ satisfies $|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$.

Definition of the Sobolev norm:

$$\|f\|_{W^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2}.$$

Hörmander (1960)

$m(D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, **if**

$$\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^n)} < \infty, \qquad s > n/2,$$

where

$$\Psi \in C_0^\infty(\mathbb{R}^n), \qquad \mathrm{supp}\,\Psi \subset \mathbb{R}^n \setminus \{0\},$$
$$\sum_{j \in \mathbb{Z}} \Psi(\xi/2^j) = 1 \qquad (\forall \xi \in \mathbb{R}^n \setminus \{0\}).$$

Calderón-Torchinsky (1977)

$m(D) : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)$, $0 < p \leq 1$, **if**

$$\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^n)} < \infty, \qquad s > n(1/p - /2).$$

It is known that these are “sharp” results; the numbers $n/2$ and $n(1/p - 1/2)$ can not be replaced by smaller numbers.

These numbers are related to “exotic” multipliers.

M (1980)

$m(D) : H^p(\mathbb{R}^n) \longrightarrow H^p(\mathbb{R}^n)$, $0 < p \leq 2$, if

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha (1 + |\xi|)^{-n(1/p-1/2)}.$$

The number $n(1/p - 1/2)$ here is also known to be sharp.

If m satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha (1+|\xi|)^{-s} \quad (\forall \alpha),$$

then it also satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

for $|\alpha| \le s$. Thus if the number $n/2$ in the theorem of Hörmander or if the number $n(1/p-1/2)$ in the theorem of Calderón-Torchinsky could be improved, then the number $n(1/p-1/2)$ in the last theorem would also be improved. But this is not the case.

We want to find the “sharp” differentiability conditions for the case of bilinear Fourier multipliers.

In this direction, the following are known.

Tomita (2010)

$T_m : L^{p_1} \times L^{p_2} \longrightarrow L^p$, $p_1, p_2, p > 1$, if

$$\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^{2n})} < \infty, \qquad s > n.$$

Grafakos–Si (preprint) extended Tomita’s result to $p \leq 1$ by using L^r-based Sobolev space with $1 < r \leq 2$.

We shall consider the problem in a different formulation. To measure the smoothness of multipliers, we use, instead of the usual Sobolev norm on $\mathbb{R}^{2n}$, the product type Sobolev norm.

Product type Sobolev norm

For $s_1,\ s_2 > 0$ and for functions $F = F(\xi_1, \xi_2)$ on $\mathbb{R}^{2n}$,

$$\|F\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}$$
$$= \left(\int_{\mathbb{R}^{2n}} (1+|x_1|)^{2s_1} (1+|x_2|)^{2s_2} |\mathcal{F}^{-1} F(x_1, x_2)|^2 \, dx_1 dx_2 \right)^{1/2}$$

We take a function Ψ such that

$$\Psi \in C_0^\infty(\mathbb{R}^{2n}), \quad \operatorname{supp} \Psi \subset \mathbb{R}^{2n} \setminus \{0\},$$
$$\sum_{j \in \mathbb{Z}} \Psi(\xi/2^j) = 1 \quad (\forall \xi \in \mathbb{R}^{2n} \setminus \{0\}).$$

For $m \in L^\infty(\mathbb{R}^{2n})$, we define

$$A_{(s_1,s_2)}(m) = \sup_{j \in \mathbb{Z}} \|m(2^j \xi_1, 2^j \xi_2) \Psi(\xi_1, \xi_2)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}$$

We shall write

$$\|T_m\|_{H^{p_1}\times H^{p_2}\to L^p}$$

to denote the smallest constant C that satisfies

$$\|T_m(f_1, f_2)\|_{L^p} \le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$$

for all $f_1 \in \mathcal{S} \cap H^{p_1}$ and $f_2 \in \mathcal{S} \cap H^{p_2}$. We define $\|T_m\|_{L^\infty\times L^\infty\to BMO}$ in the same way by replacing the norms $\|\cdot\|_{H^{p_1}}$, $\|\cdot\|_{H^{p_2}}$, $\|\cdot\|_{L^p}$ by $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{BMO}$, respectively.

Recall that $H^{p_i} = L^{p_i}$ for $1 < p_i \le \infty$.

Main results

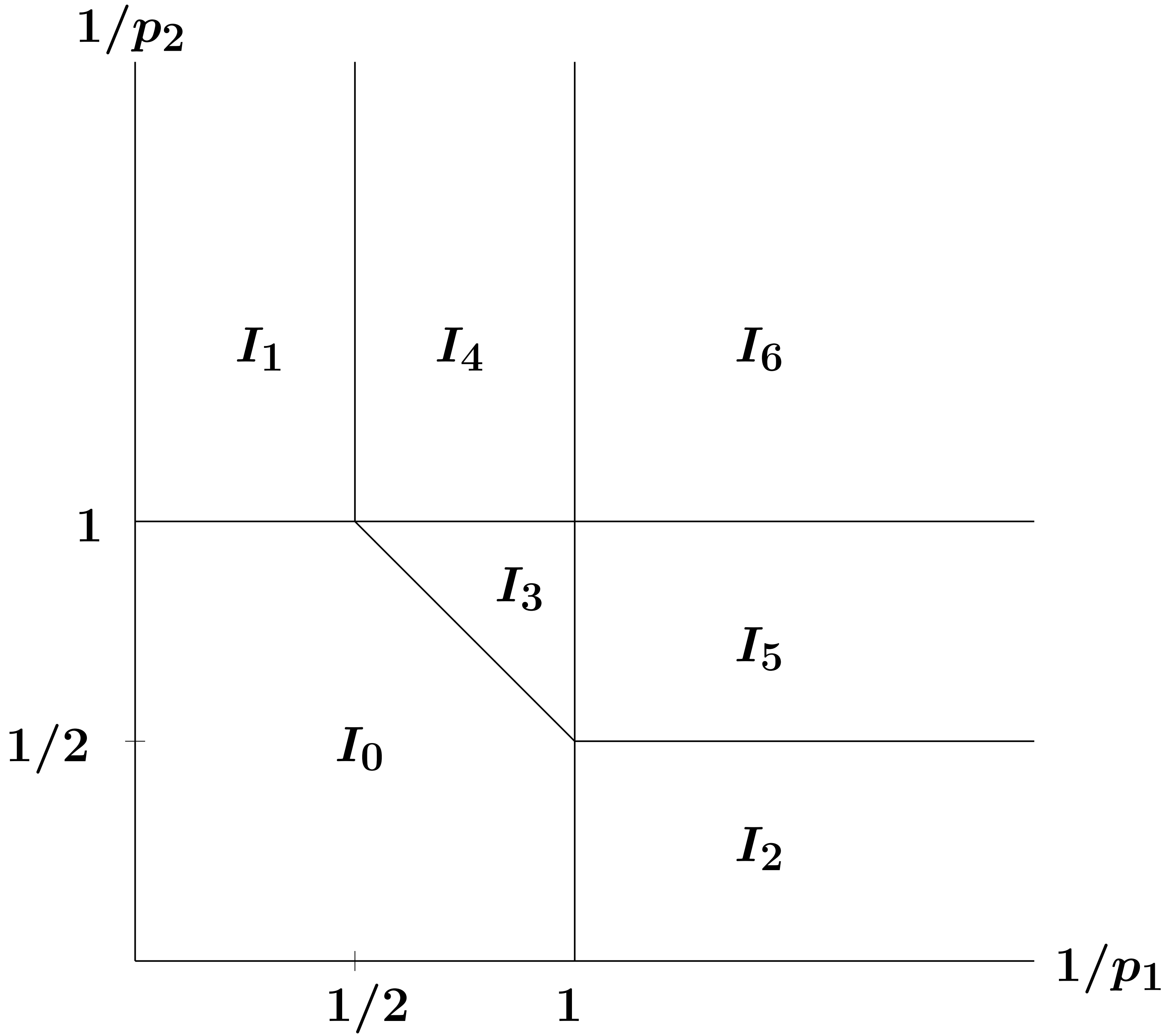
For $0 < p_1, p_2, p \le \infty$ with $1/p_1 + 1/p_2 = 1/p$, we consider the estimate

$$\|T_m\|_{H^{p_1} \times H^{p_2} \to L^p} \lesssim A_{(s_1,s_2)}(m), \tag{2}$$

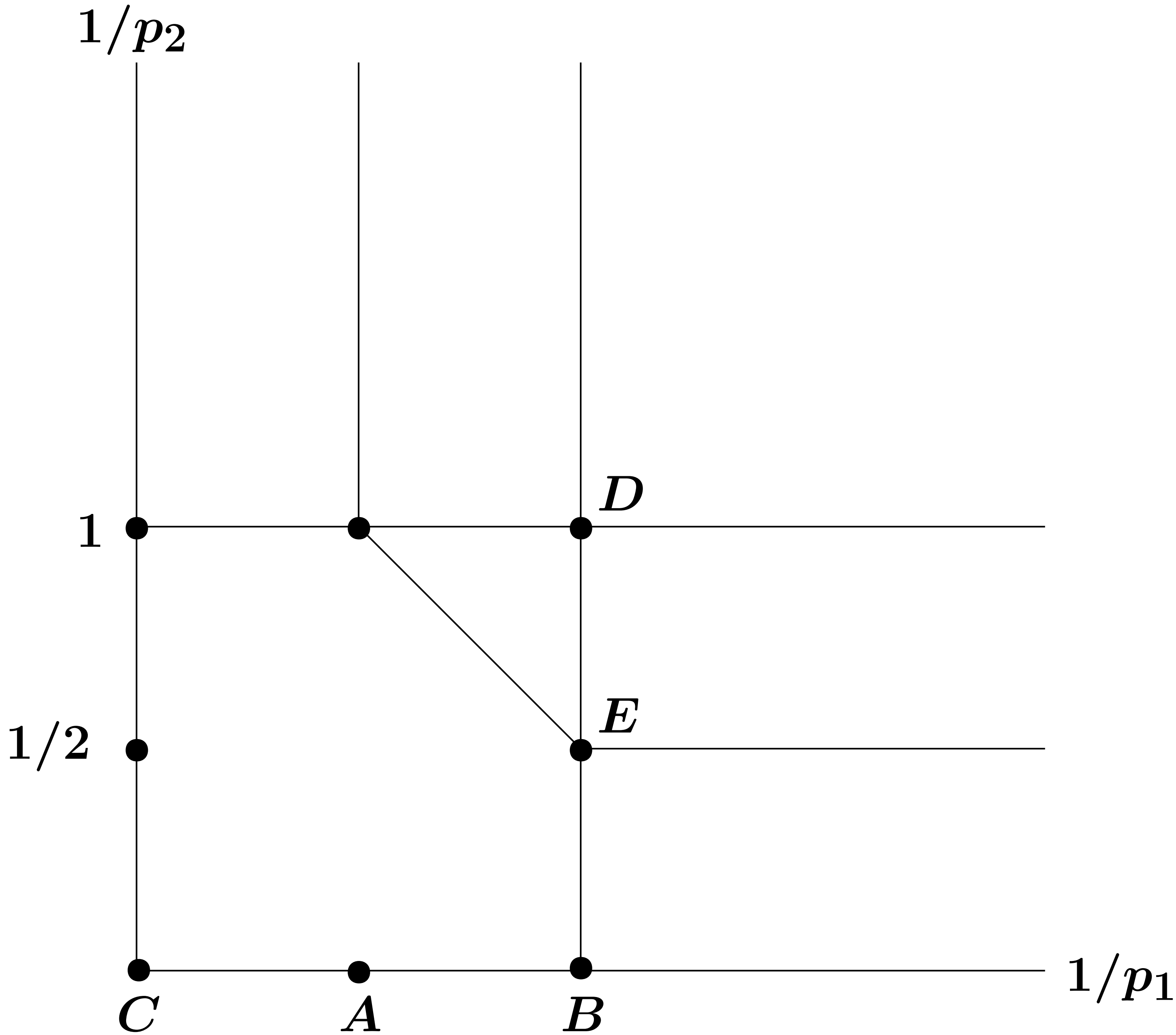
where $H^{p_1} \times H^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to BMO$ if $p_1 = p_2 = p = \infty$.

Theorem 1. The estimate (2) holds if $s_1 > \max\{n/2, n/p_1 - n/2\}$, $s_2 > \max\{n/2, n/p_2 - n/2\}$, and $s_1 + s_2 > n/p_1 + n/p_2 - n/2$.

Theorem 2. The estimate (2) holds only if $s_1 \ge \max\{n/2, n/p_1 - n/2\}$, $s_2 \ge \max\{n/2, n/p_2 - n/2\}$, and $s_1 + s_2 \ge n/p_1 + n/p_2 - n/2$.


$1/p_2$
I_1
I_4
I_6
1
I_3
I_5
$1/2$
I_0
I_2
$1/p_1$
$1/2$
1

$$
\begin{aligned}
&s_1 > n/2,\ s_2 > n/2 && \textbf{if } (1/p_1, 1/p_2) \in I_0; \\
&s_1 > n/2,\ s_2 > n/p_2 - n/2 && \textbf{if } (1/p_1, 1/p_2) \in I_1; \\
&s_1 > n/p_1 - n/2,\ s_2 > n/2 && \textbf{if } (1/p_1, 1/p_2) \in I_2; \\
&\begin{cases} s_1 > n/2,\ s_2 > n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} && \textbf{if } (1/p_1, 1/p_2) \in I_3; \\
&\begin{cases} s_1 > n/2,\ s_2 > n/p_2 - n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} && \textbf{if } (1/p_1, 1/p_2) \in I_4; \\
&\begin{cases} s_1 > n/p_1 - n/2,\ s_2 > n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} && \textbf{if } (1/p_1, 1/p_2) \in I_5; \\
&\begin{cases} s_1 > n/p_1 - n/2,\ s_2 > n/p_2 - 1/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} && \textbf{if } (1/p_1, 1/p_2) \in I_6.
\end{aligned}
$$


$1/p_2$
D
1
E
$1/2$
$1/p_1$
C
A
B

Proof of the estimate corresponding to the point A

$$\|T_m\|_{L^2 \times L^\infty \to L^2} \lesssim A_{(s_1,s_2)}(m).$$

Lemma. Let $s > n/2$. Take $\max\{1, n/s\} < q < 2$ and set $\zeta_t(x) = t^{-n}(1 + |x/t|)^{-sq}$. Suppose $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\operatorname{supp}\Psi$ is a compact subset of $\mathbb{R}^{2n} \setminus \{0\}$. Then there exists a constant $C > 0$ such that

$$\begin{aligned} &|T_{m(\cdot)\Psi(t\cdot)}(f_1, f_2)(x)| \\ &\leq C A_{(s,s)}(m)(\zeta_t * |f_1|^q)(x)^{1/q}(\zeta_t * |f_2|^q)(x)^{1/q} \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$.

Lemma. Let $b \in BMO(\mathbb{R}^n)$, $\zeta_t(x) = t^{-n}(1 + |x/t|)^{-n-\epsilon}$ with $\epsilon > 0$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\psi}(0) = 0$. Set $\psi_t(x) = t^{-1}\psi(t^{-1}x)$. Then the measure ν on $\mathbb{R}^n \times (0, \infty)$ defined by

$$d\nu = (\zeta_t * |\psi_t * b|^2)(x) \, \frac{dxdt}{t}$$

is a Carleson measure and $\|\nu\|_C \lesssim \|b\|_{BMO}^2$.

Proof of the sufficiency

$$s > n/2 \ \Rightarrow \ \|T_m\|_{L^2 \times L^\infty \to L^2} \lesssim A_{(s,s)}(m).$$

We may assume that $\operatorname{supp} m$ is contained in a cone. Suppose, for example, $\operatorname{supp} m$ is such that we have $|\xi_1| \lesssim |\xi_2| \approx |\xi_1 + \xi_2|$ for $(\xi_1, \xi_2) \in \operatorname{supp} m$.
We can take a $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ with $\widehat{\Psi}(0) = 0$ and a $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\psi}(0) = 0$ such that

$$m(\xi_1, \xi_2) = \int_0^\infty \widehat{\Psi}(t\xi_1, t\xi_2) m(\xi_1, \xi_2) \, \frac{dt}{t}$$
$$= \int_0^\infty \widehat{\Psi}(t\xi_1, t\xi_2) \widehat{\psi}(t\xi_2) \widehat{\psi}(t(\xi_1 + \xi_2)) m(\xi_1, \xi_2) \, \frac{dt}{t}.$$

Then

$$
\begin{aligned}
&T_m(f_1, f_2)(x) \\
&= \int_0^\infty \int_{\mathbb{R}^n} T_{m(\cdot)\Psi(t\cdot)}(f_1, \psi_t * f_2) * \psi_t(x)\, \frac{dxdt}{t}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\int_{\mathbb{R}^n} T_m(f_1, f_2)(x) g(x)\, dx \\
&= \int_0^\infty \int_{\mathbb{R}^n} T_{m(\cdot)\Psi(t\cdot)}(f_1, \psi_t * f_2)(x) \psi_t * g(x)\, \frac{dxdt}{t}.
\end{aligned}
$$

Using Lemmas,

$$\Big| \int_{\mathbb{R}^n} T_m(f_1, f_2)(x) g(x)\, dx \Big|$$
$$\lesssim A_{(s,s)}(m) \iint (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q}$$
$$\times |\psi_t * g(x)| \frac{dxdt}{t}$$
$$\lesssim A_{(s,s)}(m) \left(\iint (\zeta_t * |f_1|^q)(x)^{2/q} (\zeta_t * |\psi_t * f_2|^2)(x) \frac{dxdt}{t} \right)^{1/2}$$
$$\times \left(\iint |\psi_t * g(x)|^2 \frac{dxdt}{t} \right)^{1/2}$$
$$\lesssim A_{(s,s)}(m) \|f_1\|_{L^2} \|f_2\|_{BMO} \|g\|_2,$$

which impliess $\|T_m(f_1, f_2)\|_{L^2} \lesssim A_{(s,s)}(m) \|f_1\|_{L^2} \|f_2\|_{BMO}$.

Proof of the necessity

$\|T_m\|_{L^2\times L^\infty\to L^2} \lesssim A_{(s_1,s_2)}(m) \ \Rightarrow\ s_1 \geq n/2.$

We take functions ψ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\psi \neq 0, \quad \operatorname{supp} \widehat{\psi} \subset \{9/10 \leq |\xi| \leq 11/10\},$$
$$\widehat{\varphi}(0) \neq 0, \quad \operatorname{supp} \widehat{\varphi} \subset \{|\xi| \leq 1\}.$$

For sufficiently small $\epsilon > 0$, set

$$m(\xi_1,\xi_2) = m_\epsilon(\xi_1,\xi_2) = \widehat{\varphi}(\xi_1/\epsilon)\widehat{\psi}(\xi_2).$$

We have $A_{(s_1,s_2)}(m) \approx \epsilon^{-s_1+n/2}$ and

$$T_m(f_1,f_2)(x) = \mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_2}(\cdot)](x).$$

Thus the estimate $\|T_m\|_{L^2\times L^\infty\to L^2} \lesssim A_{(s_1,s_2)}(m)$ implies

$$
\begin{aligned}
&\|\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_2}(\cdot)](x)\|_{L^2} \\
&\lesssim \epsilon^{-s_1+n/2}\|f_1\|_{L^2}\|f_2\|_{L^\infty}.
\end{aligned}
$$

We test this inequality for

$$
\widehat{f_1}(\xi) = \epsilon^{-n/2}\widehat{\varphi}(\xi_1/\epsilon), \qquad f_2(x) = e^{i\eta^\circ x},
$$

where we choose η° so that $\widehat{\psi}(\eta^\circ) \neq 0$. Then we obtain $1 \lesssim \epsilon^{-s_1+n/2}$ and thus $s_1 \geq n/2$.

Proof of the estimate corresponding to the point B

$$\|T_m(f_1, f_2)\|_{L^1} \lesssim A_{(s_1,s_2)}(m)\|f_1\|_{H^1}\|f_2\|_{L^\infty}$$

for $s_1 > n/2$ and $s_2 > n/2$.

To prove this it is sufficient to show

$$\|T_m(f_1, f_2)\|_{L^1} \lesssim A_{(s_1,s_2)}(m)\|f_2\|_{L^\infty}$$

for all H^1 atoms f_1. This can be done in the same way as in the case of linear multiplier operator.

Proof of the estimate corresponding to the point C

$$\|T_m\|_{L^\infty \times L^\infty \to BMO} \lesssim A_{(s_1,s_2)}(m)$$

for $s_1 > n/2$ and $s_2 > n/2$.

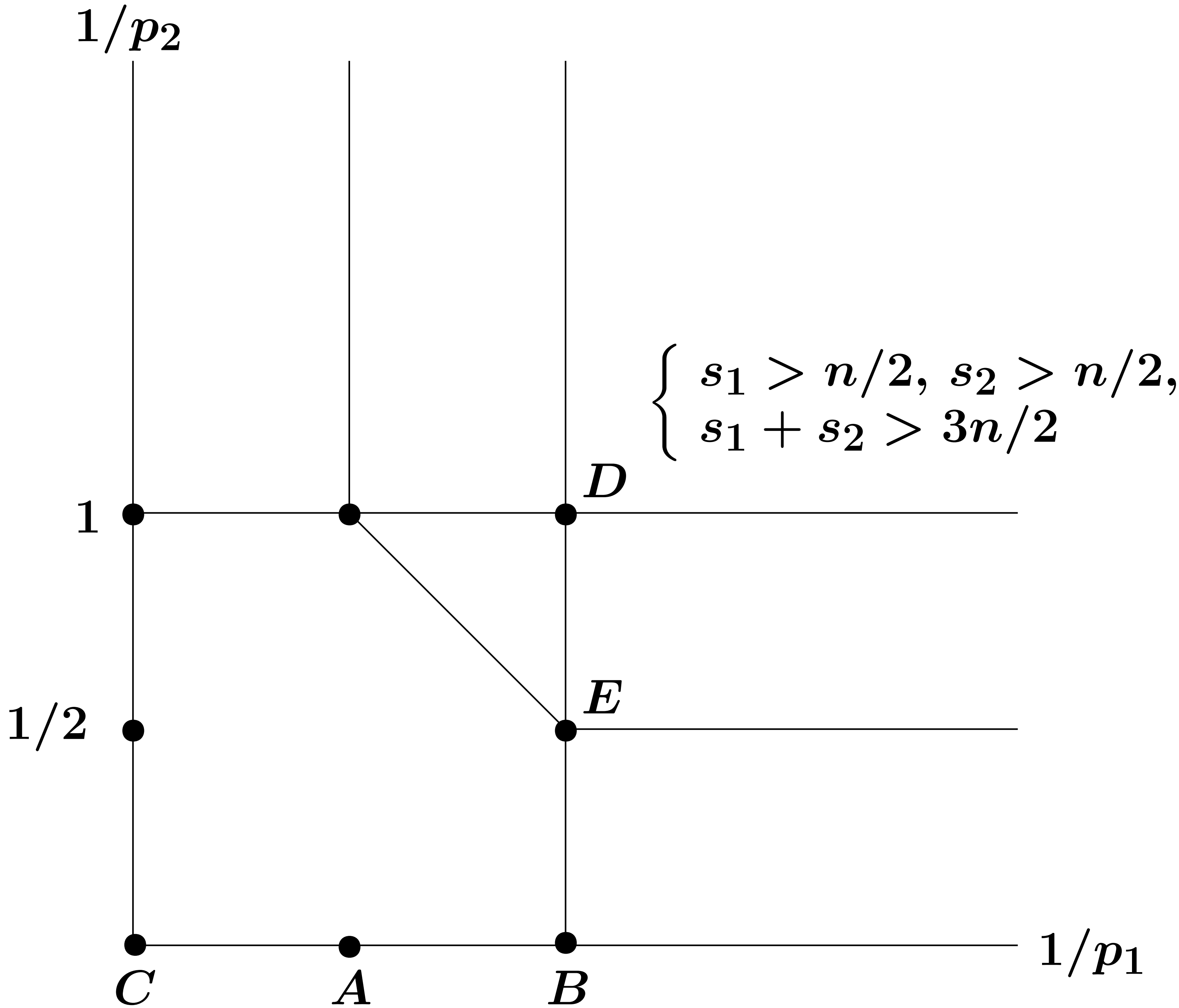
This can be done by proving that the kernel $K = \mathcal{F}^{-1}m$ satisfies the following variant of the Hörmander condition: If $s_1, s_2 > n/2$, then

$$\int_{\substack{|y_1|>2|x| \\ |y_2|>2|x|}} |K(x+y_1, x+y_2) - K(y_1,y_2)|\, dy_1 dy_2 \lesssim A_{(s_1,s_2)}(m).$$

The results corresponding to the point D

$$s_1 > n/2,\ s_2 > n/2,\ s_1 + s_2 > 3n/2$$
$$\Rightarrow\ \|T_m\|_{H^1 \times H^1 \to L^{1/2}} \lesssim A_{(s_1,s_2)}(m).$$

$$\|T_m\|_{H^1 \times H^1 \to L^{1/2}} \lesssim A_{(s_1,s_2)}(m)$$
$$\Rightarrow\ s_1 \geq n/2,\ s_2 \geq n/2,\ s_1 + s_2 \geq 3n/2.$$


$1/p_2$
$\begin{cases} s_1 > n/2,\ s_2 > n/2, \\ s_1 + s_2 > 3n/2 \end{cases}$
D
1
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$1/p_1$
C
A
B

Proof of the necessity

$$\|T_m\|_{H^1\times H^1\to L^{1/2}} \lesssim A_{(s_1,s_2)}(m) \ \Rightarrow\ s_1+s_2 \geq 3n/2.$$

Recall that

$$T_m(f_1,f_2)(x) \\ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y_1,x-y_2)f_1(y_1)f_2(y_2)\,dy_1dy_2,$$

where $K = \mathcal{F}^{-1}m$. Thus the behavior of the kernel $K(x_1,x_2)$ along the diagonal $x_1 = x_2$ is important for $T_m(f_1,f_2)(x)$. In order to get multiplier whose inverse Fourier transform decays slowly along the diagonal, we consider the following multiplier.

Take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

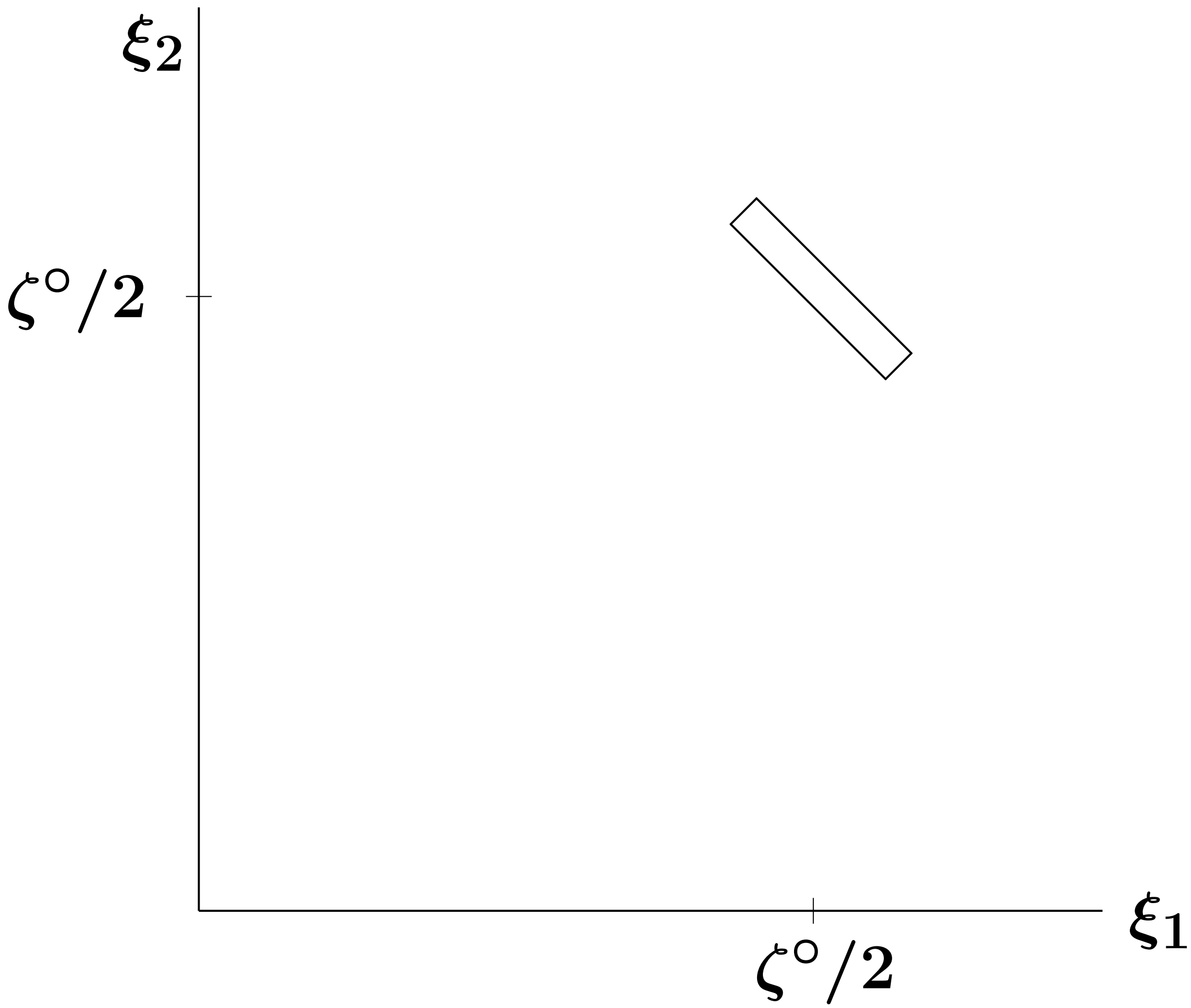
$$\operatorname{supp} \widehat{\varphi} \subset \{|\xi| \leq 1/10\}, \qquad \widehat{\varphi}(\xi) \geq 0,$$

$$\widehat{\varphi}(\xi) = 1 \quad \text{if} \quad |\xi| \leq 1/20.$$

Take a point $\zeta^\circ \in \mathbb{R}^n$ satisfying $|\zeta^\circ| = \sqrt{2}$, and set, for sufficiently small $\epsilon > 0$,

$$m^{(\epsilon)}(\xi_1, \xi_2) = \widehat{\varphi}\left(\frac{\xi_1 + \xi_2 - \zeta^\circ}{\epsilon}\right) \widehat{\varphi}(\xi_1 - \xi_2).$$

$\operatorname{supp} m^{(\epsilon)}$ **is contained in such a rectangle:**



It can be shown that

$$A_{(s_1,s_2)}(m^{(\epsilon)}) \approx \epsilon^{n/2-s_1-s_2}.$$

We take $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{f}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$ and $\widehat{f}(\xi_1)\widehat{f}(\xi_2) = 1$ on $\operatorname{supp} m^{(\epsilon)}$. Then

$$\begin{aligned} T_{m^{(\epsilon)}}(f,f)(x) &= \mathcal{F}^{-1}(m^{(\epsilon)})(x,x) \\ &= c \exp(i\zeta^{\circ} x)\epsilon^n \varphi(\epsilon x)\varphi(0). \end{aligned}$$

Hence the estimate $\|T_m\|_{H^1 \times H^1 \to L^{1/2}} \lesssim A_{(s_1,s_2)}(m)$ for $m = m^{(\epsilon)}$ implies

$$\| \exp(i\zeta^{\circ} x)\epsilon^n \varphi(\epsilon x)\varphi(0)\|_{L^{1/2}} \lesssim \epsilon^{n/2-s_1-s_2}$$

or $\epsilon^{-n} \lesssim \epsilon^{n/2-s_1-s_2}$ and thus $s_1 + s_2 \geq 3n/2$.

Proof of the sufficiency at the point D

$$s_1 > n/2,\ s_2 > n/2,\ s_1 + s_2 > 3n/2$$
$$\Rightarrow\ \|T_m\|_{H^1\times H^1\to L^{1/2}} \lesssim A_{(s_1,s_2)}(m).$$

We decompose f_1 and f_2 into H^1-atoms:

$$f_1 = \sum_j \lambda_{1,j} a_{1,j}, \qquad \sum_j |\lambda_{1,j}| \approx \|f_2\|_{H^1},$$
$$f_2 = \sum_k \lambda_{2,k} a_{2,k}. \qquad \sum_k |\lambda_{1,k}| \approx \|f_2\|_{H^1},$$

Then $\quad T_m(f_1, f_2) = \sum_{j,k} \lambda_{1,j}\lambda_{2,k} T_m(a_{1,j}, a_{2,k}).$

Notice that it is not sufficient to show

$$\|T_m(a_{1,j}a_{2,k})\|_{L^{1/2}} \le c,$$

since the subadditivity of $\|\cdot\|_{L^{1/2}}^{1/2}$ implies only

$$\begin{aligned}
&\Big\|\sum_{j,k}\lambda_{1,j}\lambda_{2,k}T_m(a_{1,j},a_{2,k})\Big\|_{L^{1/2}}^{1/2} \\
&\le c^{1/2}\sum_{j,k}|\lambda_{1,j}|^{1/2}|\lambda_{2,k}|^{1/2} \\
&= c^{1/2}\Big(\sum_{j}|\lambda_{1,j}|^{1/2}\Big)\Big(\sum_{k}|\lambda_{2,k}|^{1/2}\Big),
\end{aligned}$$

which can not be controlled.

Here is a Lemma that can be applied to general bilinear operators.

Lemma A. Let T be a continuous bilinear operator $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n)$ and let $0 < p_1, p_2 < \infty$, $0 < p \le 1$, $1/p_1 + 1/p_2 = 1/p$. Suppose

$$\|T(a_1, a_2)\|_{L^p}^p \le \|a_1\|_{L^\infty}^p \|a_2\|_{L^\infty}^p |Q_2| \left(1 + \frac{|c(Q_1) - c(Q_2)|}{\ell(Q_1)}\right)^{-n-\epsilon}$$

whenever $a_j \in L^\infty$, $\operatorname{supp} a_j \subset Q_j$, $\int a_j(x) x^\alpha dx = 0$ for $|\alpha| \le N_j - 1$, and $\ell(Q_1) \ge \ell(Q_2)$. Suppose also the similar estimate for the case $\ell(Q_1) \le \ell(Q_2)$. Then $\|T(f_1, f_2)\|_{L^p} \le c\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$.

But this general lemma alone does not seem to give the sharp result for $T_m : H^1 \times H^1 \to L^{1/2}$.

Let $Q_{1,j}$ and $Q_{2,k}$ be the supporting cubes the atoms $a_{1,j}$ and $a_{2,k}$, and let $Q^*_{1,j} = aQ_{1,j}$ and $Q^*_{2,k} = aQ_{2,k}$ with a sufficiently large.

We decompose the sum into two parts,

$$\mathrm{I} = \sum_{j,k} \lambda_{1,j}\lambda_{2,k} T_m(a_{1,j}, a_{2,k}) 1_{\mathrm{Q}^*_{1,j} \cap \mathrm{Q}^*_{2,k}},$$
$$\mathrm{II} = \sum_{j,k} \lambda_{1,j}\lambda_{2,k} T_m(a_{1,j}, a_{2,k}) 1_{(\mathrm{Q}^*_{1,j} \cap \mathrm{Q}^*_{2,k})^{\mathrm{c}}}.$$

The part I can be treated by the use of L^2-estimate for T_m and Lemma A.

In order to estimate the part II, we prove the pointwise estimate

$$|T_m(a_{1,j}, a_{2,k})(x)| \lesssim b_{1,j}(x) b_{2,k}(x) \text{ for } x \in (Q^*_{1,j} \cap Q^*_{2,k})^c,$$
$$\|b_{1,j}\|_{L^1} \lesssim 1, \quad \|b_{2,k}\|_{L^1} \lesssim 1.$$

Using this pointwise estimate, we have

$$\begin{aligned} |\mathrm{II}| &\lesssim \sum_{j,k} |\lambda_{1,j}||\lambda_{2,k}| b_{1,j}(x) b_{2,k}(x) \\ &= \Big(\sum_j |\lambda_{1,j}| b_{1,j}(x) \Big) \Big(\sum_k |\lambda_{2,k}| b_{2,k}(x) \Big) \end{aligned}$$

and thus $\|\mathrm{II}\|_{L^{1/2}} \lesssim \|f_1\|_{H^1} \|f_2\|_{H^1}$.

Notice that our pointwise estimate should be of the form

$$|T_m(a_{1,j}, a_{2,k})(x)| \lesssim b_{1,j}(x) b_{2,k}(x),$$

not of the form

$$|T_m(a_{1,j}, a_{2,k})(x)| \lesssim b_{1,j,k}(x) b_{2,j,k}(x).$$

Proof of the estimate corresponding to point E

$$s_1 > n/2,\ s_2 > n/2$$
$$\Rightarrow\ \|T_m\|_{H^1\times L^2\to L^{2/3}} \lesssim A_{(s_1,s_2)}(m).$$

This can be proved by the method similar to the case for point D.

Multilinear Fourier multipliers

For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$
\begin{aligned}
&T_m(f_1, \cdots, f_N)(x) \\
&= \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix(\xi_1 + \cdots + \xi_N)} \\
&\qquad\qquad m(\xi_1, \cdots, \xi_N) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N)\, d\xi_1 \cdots d\xi_N,
\end{aligned}
$$

where $f_1, \cdots, f_N \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

$$A_{(s_1,\cdots,s_N)}(m)$$
$$= \sup_{j\in\mathbb{Z}} \|m(2^j\xi_1,\cdots,2^j\xi_N)\Psi(\xi_1,\cdots,\xi_N)\|_{W^{(s_1,\cdots,s_N)}(\mathbb{R}^{Nn})}.$$

where $\|\cdot\|_{W^{(s_1,\cdots,s_N)}(\mathbb{R}^{Nn})}$ is the product type Sobolev norm.

Theorem 3. If $s_1, \cdots, s_N > n/2$, then

$$\|T_m\|_{L^2\times L^\infty\times\cdots\times L^\infty\to L^2} \lesssim A_{(s_1,\cdots,s_N)}(m). \tag{3}$$

Theorem 3′. The estimate (3) holds only if $s_1, s_2, \cdots, s_N \geq n/2$.

Theorem 4. If $0 < p \leq 1$ and if $s_1 > n(1/p - 1/2)$ and $s_2, \cdots, s_N > n/2$, then

$$\|T_m\|_{H^p \times L^\infty \times \cdots \times L^\infty \to L^p} \lesssim A_{(s_1, s_2, \cdots, s_N)}(m). \qquad (4)$$

Theorem 4′. For $0 < p \leq 1$, the estimate (4) holds only if $s_1 \geq n(1/p - 1/2)$ and $s_2, \cdots, s_N \geq n/2$.

Further results and problems

(1) Improve the estimate in the pentagon region.

For the linear multiplier operator $m(D)$, we have the obvious the L^2-estimate $\|m(D)\|_{L^2 \to L^2} \leq \|m\|_{L^\infty}$; we don't need any differentiability for m.

Are there similar estimate for T_m?

(2) Use of the usual Sobolev scale.

Tomita (2010): $T_m : L^{p_1} \times L^{p_2} \to L^p$, $p_1, p_2, p > 1$, **if**

$$\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^{2n})} < \infty, \qquad s > n.$$

Can we extend this to the case $1 \geq p > 1/2$**?**

Grafakos–Si recently generalized Tomita's result to $1 \geq p > 1/2$ **by using the** L^r**-based Sobolev space with** $1 < r \leq 2$.

Our result shows Tomita's result can be extended to $1 \geq p > 2/3$.

(3) Replace $H^p \times L^\infty \longrightarrow L^p$ by $H^p \times BMO \longrightarrow L^p$.

If we assume conditions as $m(\xi_1, 0) = 0$, then $L^2 \times L^\infty \longrightarrow L^2$ can be improved to $L^2 \times BMO \longrightarrow L^2$.

(4) Replace $H^{p_1} \times H^{p_2} \longrightarrow L^p$ by $H^{p_1} \times H^{p_2} \longrightarrow H^p$.

If we assume $m(\xi_1, \xi_2) = 0$ for $\xi_1 + \xi_2 = 0$, then $L^{p_1} \times L^{p_2} \longrightarrow L^1$ can be improved to $L^{p_1} \times L^{p_2} \longrightarrow H^1$.