One–weighted and Trace Inequalities Criteria for Fractional Integrals in grand Lebesgue spaces

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Grand Lebesgue spaces

Nowadays the theory of grand Lebesgue spaces is one of the intensively developing directions of the modern analysis. It was realized the necessity for the study of these spaces because of their rather essential role and applications in various fields. These spaces naturally arise, for example, in the integrability problems of the Jacobian under minimal hypothesis (see [T. Iwaniec and C. Sbordone, 1992]).

We are focused on singular and fractional integral operators is these spaces from the boundedness viewpoint.

Grand Lebesgue spaces

The talk consists of three sections. In Section 1 we discuss that the boundedness of the fractional integral operator I_α between two grand Lebesgue spaces $L^{p),\theta_1}$ and $L^{q),\theta_2}$ ($q = p/(1-\alpha p)$) depends on the choice of the second parameters θ_1 and θ_2. Moreover, we determine the sets of pairs (θ_1, θ_2) for which this boundedness holds; we present one–weight criteria for fractional integral operator with product kernels.
In Section 2 we show that the trace inequality for fractional integral operators defined on metric measure spaces holds if and only if D. Adams–type condition is satisfied.
In Section 3 we discuss that the one–weight inequality for the Hardy transform in grand Lebesgue spaces for non-increasing functions holds if and only if the well–known Arino–Muckenhoupt condition is satisfied. Moreover, we apply this result to establish boundedness criteria for the Hardy–Littlewood maximal operator in grand Lorentz spaces.

Grand Lebesgue spaces. Definition

Let G be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$ and $\theta > 0$. Suppose that w is an integrable a.e. positive function (weight) on G. We denote by $L^{p),\theta}_w$ weighted generalized grand Lebesgue space. It is a Banach function space with respect to the norm

$$\|f\|_{L^{p),\theta}_w} = \sup_{0<\varepsilon\le p-1} \left(\frac{\varepsilon^\theta}{|G|} \int_G |f(x)|^{p-\varepsilon} w(x)dx \right)^{\frac{1}{p-\varepsilon}}.$$

If $w =$ *const*, then we denote this space by $L^{p),\theta}$. If $\theta = 1$, then we have Iwaniec-Sbordone space $L^{p)}$. Structural properties of the spaces $L^{p),\theta}$ were studied by A. Fiorenza; C. Capone and A. Fiorenza etc.

Grand Lebesgue spaces. Properties

The space $L^{p),\theta)}$ is a rearrangement invariant Banach function space.
The spaces $L^{p),\theta}$ for $\theta > 0$ were introduced by L. Greco, T. Iwaniec and C. Sbordone (1997) when they studied the existence and uniqueness of the solution of the equation

$$div|\nabla u|^{n-2}\nabla u = \mu.$$

Grand Lebesgue spaces. Properties

It is easy to see that the following continuous embeddings hold:

$$L^p_w \subset L^{p),\theta_1}_w \subset L^{p),\theta_2}_w \subset L^{p-\varepsilon}_w, \quad 0 < \varepsilon \le p-1,\ \theta_1 < \theta_2.$$

(a) the norm $\|\cdot\|_{L^{p),\theta}}$ is not absolutely continuous norm;
(b) $L^{p),\theta}$ is not reflexive;
(c) the closure of L^p in $L^{p),\theta}$ denoted by $\mathcal{L}^{p),\theta}$, does not coincide with $L^{p),\theta}$. Moreover, for $f \in \mathcal{L}^{p),\theta}$, one has

$$\lim_{\varepsilon\to 0} \varepsilon^\theta \int_G |f(x)|^{p-\varepsilon} dx = 0.$$

(d) in general, the weighted space $L^{p),\theta}_w$ is not rearrangement invariant.

Weighted Grand Lebesgue spaces. Muckenhoupt Class

The general theory of rearrangement invariant Banach Function Spaces can not be used in grand Lebesgue spaces. For example, Lorentz–Shimogaki (For the Hardy–Littlewood maximal operator) and Boyd (for the Hilbert transform) theorems are not suitable to these spaces because these spaces are defined on sets of finite measure.
In the classical weighted Lebesgue spaces the equality

$$\|f\|_{L^p_w} = \|w^{1/p} f\|_{L^p}$$

holds. This means that

$$f \in L^p_w \Leftrightarrow w^{1/p} f \in L^p.$$

In grand Lebesgue spaces we **do not** have such a property: there is a function $f \in L^{p)}_w$ such that $w^{1/p} f \notin L^{p)}$

Weighted Grand Lebesgue spaces. Maximal Operator and Hilbert Transform

For the simplicity let us assume that $G = [0,1]$.
Let M be the Hardy–Littlewood maximal operator defined on $[0,1]$:

$$(Mf)(x) = \sup_{J \ni x} \frac{1}{|J|} \int_J |f(t)| dt,$$

where the supremum is taken over all subintervals J of $[0,1]$ containing x.
Denote by H the Hilbert transform on $[0,1]$:

$$Hf(x) = p.v. \int_0^1 \frac{f(t)}{x-t} dt; \quad x \in [0,1].$$

It is known that (see A. Fiorenza, B. Gupta and P. Jain, 2008) the operator M is bounded in $L_w^{p)}$ if and only if $w \in A_p$, i.e.

$$\sup_{I\subset[0,1]} \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{1-p'}\right)^{p-1} < \infty, \quad p' = \frac{p}{p-1}.$$

It was shown by V. Kokilashvili and A.Meskhi that the Hilbert transform H is bounded in $L_w^{p)}$ if and only if $w \in A_p$.

Let R_0 be a fixed rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes and let

$$(T_\alpha f)(x) = \int\limits_{R_0} \frac{f(t_1, \cdots, t_n)}{\prod_{i=1}^n |x_i - t_i|^{1-\alpha}} dt_1 \cdots dt_n, \quad x = (x_1, \cdots, x_n) \in R_0,$$

be the fractional integral operator with product kernels, where $0 < \alpha < 1$. In this case potential kernels **have singularity not only at the origin but on the hyperplanes**.

Related fractional maximal operator is given by

$$(M_\alpha f)(x) = \sup_{R \ni x} \frac{1}{|R|^{1-\alpha}} \int\limits_R |f(t)| dt, \quad x \in R_0,$$

where the supremum is taken over all subrectangles R (with sides parallel to coordinate exes) of R_0 containing x.

Grand Lebesgue spaces. Fractional Integrals in Classical Lebesgue Spaces

If $n = 1$, then we have the single potential operator

$$(I_\alpha f)(x) = \int_I \frac{f(t)}{|x-t|^{1-\alpha}} dt, \quad 0 < \alpha < 1.$$

I_α (consequently, T_α or M_α) is bounded from L^p to L^q, where $1 < p < \frac{1}{\alpha}$ and $p \leq q \leq \frac{p}{1-\alpha p}$.

Grand Lebesgue spaces. Fractional Integrals in Unweighted Grand Lebesgue Spaces

We have noticed that the boundedness of the fractional integral operator **depends on the choice of the second parameter** of the space. In fact this was first time when we encountered this situation. In particular, the following statement holds:

Theorem 1. *Let $n = 1$. Let $1 < p < 1/\alpha$. We set $q = \frac{p}{1-\alpha p}$.*

(i) *The operator I_α* **is not bounded** *from $L^{p),\theta_1}$ to $L^{q),\theta_2}$, where $\theta_2 < \frac{\theta_1 q}{p}$; in particular, I_α* **is not bounded** *from $L^{p),\theta}$ to $L^{q),\theta}$;*

(ii) *The operator I_α* **is bounded** *from $L^{p),\theta}$ to $L^{q),\frac{q}{p}\theta}$ (hence it* **is bounded** *also from $L^{p),\theta_1}$ to $L^{q),\theta_2}$ for $\theta_2 > \frac{\theta_1 q}{p}$).*

Grand Lebesgue spaces. Weighted Criteria in Lebesgue Spaces

Let R_0 be a rectangle with sides parallel to coordinate exes. We say that the weight function w belongs to the class $\mathcal{A}_p$, $p > 1$, if

$$\sup_{R \subset R_0} \left(\frac{1}{|R|} \int_R w \right)^{1/p} \left(\frac{1}{|R|} \int_R w^{1-p'} \right)^{1/p'} < \infty,$$

where the "sup" is taken over all $n-$ dimensional subintervals R of R_0. It is known that (see B. Muckenhoupt and R.L. Wheeden, 1974 for $n = 1$, V. Kokilashvili, 1984 for $n > 1$) that the one–weight inequality

$$\| T_\alpha (f w^\alpha) \|_{L^q_w} \leq c \| f \|_{L^p_w},$$

for the multiple potential operator T_α, where w is a weight on R_0, holds if and only if $w \in \mathcal{A}_{1+q/p'}(R_0)$, i.e.

Grand Lebesgue spaces. Fractional Integrals. One–weight Criteria in Grand Lebesgue Spaces

Theorem 2. *Let $n \geq 1$. Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that $\theta > 0$. We set $q = \frac{p}{1-\alpha p}$. Then the following conditions are equivalent:*

$$(i) \quad \|T_\alpha(fw^\alpha)\|_{L_w^{q),\theta q/p}(R_0)} \leq c\|f\|_{L_w^{p),\theta}(R_0)},$$

$$(ii) \quad \|M_\alpha(fw^\alpha)\|_{L_w^{q),\theta q/p}([0,1])} \leq c\|f\|_{L_w^{p),\theta}(R_0)},$$

$(iii) \qquad w \in \mathcal{A}_{1+q/p'}(R_0).$

Now we discuss the trace inequality for fractional integrals. We derived trace inequality criteria for potential operators defined on metric measure spaces.

Let (X, d, μ) be a metric measure space with finite measure μ and let

$$(P_\alpha f)(x) = \int_X \frac{f(y)}{\mu(B(x, d(x,y)))^{1-\alpha}} d\mu(y), \quad 0 < \alpha < 1$$

be the fractional integral operator on X, where by the symbol $B(x, r)$ is denoted the ball with center x and radius r.

We denote by $L^{p),\theta}(X, \mu)$ the grand Lebesgue space defined with respect to μ.

Definition. Let (X, d, μ) be a metric measure space. We say that μ satisfies the doubling condition ($\mu \in DC(X)$) if there is a positive constant b such that for all $x \in X$ and $r > 0$ the inequality

$$\mu B(x, 2r) \leq b \mu B(x, r)$$

holds.

Examples of SHT

A metric measure space (X, d, μ) with doubling measure μ is called a space of homogeneous type (SHT).

Example of an SHT with finite measure is a bounded $s-$ set Γ of $\mathbb{R}^n$ $(0 < s \le n)$ in the sense that there is a Borel measure μ on $\mathbb{R}^n$ such that

(i) supp $\mu = \Gamma$;

(ii) there are positive constants c_1 and c_2 such that for all $x \in \Gamma$ and all $r \in (0, \text{diam } \Gamma)$,

$$c_1 r^s \le \mu\big(\Gamma(x, r)\big) \le c_2 r^s, \qquad (*)$$

where $\Gamma(x, r) := B(x, r) \cap \Gamma$ and $B(x, r)$ is a ball in $\mathbf{R}^n$ with center x and radius r.

It is known that μ is equivalent to the restriction of Hausdorff $s-$ measure $\mathcal{H}_s$ to Γ; we shall identify μ with $\mathcal{H}_s|_\Gamma$.

For example, connected rectifiable regular curves with respect to the arc–length measure satisfy condition $(*)$ for $s = 1$.

2. Grand Lebesgue spaces. Fractional Integrals. Trace Inequality

The next statement is due to D. Adams (1972) for Euclidean spaces with the Lebesgue measure and M. Gabidzashvili (1986) for an SHT (see also the monograph D. E. Edmunds, V.Kokilashvili and A. Meskhi, Bounded and compact integral operators, *Kluwer, Dordrecht*, 2002, Ch.6).

Theorem A. *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) be an SHT. Suppose that there is defined another measure ν on X. Then the operator P_α is bounded from $L^p(X, \mu)$ to $L^q(X, \nu)$ if and only if there is a positive constant C such that for all balls $B \subset X$ the inequality*

$$\nu(B) \leq C\mu(B)^{q(1/p-\alpha)} \tag{1}$$

holds.

Grand Lebesgue spaces. Fractional Integrals. Trace Inequality

For grand Lebesgue spaces we proved

Theorem 4. *Let* $1 < p < q < \infty$ *and let* $0 < \alpha < 1/p$. *Suppose that* (X, d, μ) *be an SHT with finite measure* μ. *Suppose that there is defined another finite measure* ν *on* X. *Then the operator* P_α *is bounded from* $L^{p),\theta}(X, \mu)$ *to* $L^{q),q\theta/p}(X, \nu)$ *if and only if* (1) *holds.*

First we prove the general–type result from which it follows Theorem 4. To formulate this result we need some definitions.

Let (X, μ) be a finite measure space. Let $1 < p < \infty$ and let φ be a continuous positive function on $(0, p-1)$ satisfying the condition $\lim\limits_{x \to 0+} \varphi(x) = 0$. The generalized grand Lebesgue spaces $L^{p),\varphi(\cdot)}(X, \mu)$ is the class of those $f : X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi(\cdot)}(X,\mu)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varphi(\varepsilon)}{\mu(X)} \int\limits_X |f(x)|^{p-\varepsilon} d\mu(x) \right)^{1/(p-\varepsilon)}$$

is finite.

If $\varphi(x) = x^{\theta}$, where θ is a positive number, then we have the grand Lebesgue space $L^{p),\theta}(X, \mu)$.

Definition 1. *Let $1 < p < q < \infty$. Suppose that $M_{p,q}(X, Y)$ is a class of pairs (μ, ν) of finite measures, where (X, d, μ) and (Y, ρ, ν) are quasi-metric measure spaces. We say that a linear operator T belongs to the class $\mathcal{B}(M_{p,q}(X, Y))$ if T is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$ for every $(\mu, \nu) \in M_{p,q}(X, Y)$.*

Definition 2. Let $1 < p < q < \infty$. We say that a class $M_{p,q}(X, Y)$ of couples of finite measures (μ, ν), where (X, d, μ) and (Y, ρ, ν) are metric measure spaces, is allowable if the following condition is satisfied:
$(\mu, \nu) \in M_{p,q}(X, Y) \Rightarrow (\mu, \nu) \in M_{p-\eta_0, q-\varepsilon_0}(X, Y)$ for some numbers $\varepsilon_0 \in (0, q-1)$, $\eta_0 \in (0, p-1)$.

If $X = Y$, then we denote $M_{p,q}(X, Y)$ by $M_{p,q}(X)$.

Grand Lebesgue spaces. Fractional Integrals. Trace Inequality

Let $1 < r < \infty$. We denote by P_r the class of those continuous functions $\phi : [0, r-1) \to (0, \infty)$ satisfying the condition $\lim_{x \to 0} \phi(x) = 0$.

Let $1 < p < q < \infty$ and let ε_0 and η_0 satisfy the conditions: $0 < \varepsilon_0 < p - 1$, $0 < \eta_0 < q - 1$. Let us introduce the functions:

$$g(\eta) := \frac{\eta q \varepsilon_0 (p - \eta_0)}{\eta_0 (q - \varepsilon_0)(p - \eta) + \eta \varepsilon_0 (p - \eta_0)};$$

$$\Psi(x) := \Big[\Phi(g(x))\Big]^{\frac{p-x}{q-g(x)}}, \tag{2}$$

where $\Phi \in P_q$. Observe that $g, \Psi \in P_p$ and that $g(\eta_0) = \varepsilon_0$.

Grand Lebesgue spaces. General Result

Theorem 5 (General Result). *Let $1 < p < q < \infty$ and let $M_{p,q}(X, Y)$ be allowable class of pairs of finite measures with constants ε_0 and η_0 . Assume that*

$$T \in \mathcal{B}(M_{p,q}(X, Y)) \cap \mathcal{B}(M_{p-\eta_0, q-\varepsilon_0}(X, Y)).$$

Then T is bounded from $L^{p),\Psi(\cdot)}(X, \mu)$ to $L^{q),\Phi(\cdot)}(Y, \nu)$ for $(\mu, \nu) \in M_{p,q}(X, Y)$, where Ψ and Φ are related by (2).

3. Hardy Transform under the B_p Condition

In this section we show that for the boundedness of the Hardy operator

$$Hf(x) = \frac{1}{x}\int\limits_0^x f(t)dt$$

from $L^{p),\theta}_{dec,w}(I)$ to $L^{p),\theta}_{w}(I)$ $\big(0 < p < \infty, \quad \theta > 0, \ I = (0,1) \ \big)$ it is necessary and sufficient that the weight w belongs to the well–known class B_p restricted to the interval I. This result is applied to derive the boundedness of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int\limits_Q f(y)dy, \quad x \in \mathbf{R}^n,$$

where the supremum is taken over all cubes Q containing x, in the weighted grand Lorentz space $\Lambda^{p),\theta}_{w}$.

Grand Lebesgue Space $L_w^{p),\theta}$ for $0 < p < \infty$

Let $0 < p < \infty$ and let $\theta > 0$. We will assume that $I := (0,1)$. Suppose that w is integrable a.e. non–negative function (i.e. weight) on I. It is assumed that

$$\int_0^r w(x)dx > 0, \quad \text{for all } r \in I.$$

We denote by $L_w^{p),\theta}(I)$ the generalized grand Lebesgue space. This is the class of all measurable functions $f : I \to \mathbf{R}$ for which

$$\|f\|_{L_w^{p),\theta}(I)} := \sup_{0<\varepsilon\le\varepsilon_0} \left(\varepsilon^\theta \int_0^1 |f(t)|^{p-\varepsilon} w(t)dt \right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $\varepsilon_0 = p - 1$ if $p > 1$; $\varepsilon_0 \in (0, p)$ if $0 < p \le 1$.

B_p Condition

For the next statement we refer to [Arino and Muckenhoupt, 1990]:

Theorem B. *Let* $0 < p < \infty$ *and let* w *be non–negative function on* $\mathbf{R}_+$*. Then the inequality*

$$\int_0^\infty \big(Hf(x)\big)^p w(x)dx \le c \int_0^\infty (f(x))^p w(x)dx$$

holds for all non–negative and non–increasing functions f *on* $\mathbf{R}_+$ *if and only if* $w \in B_p$*, i.e., there is a positive constant* B *such that for all* $r > 0$,

$$r^p \int_r^\infty \frac{w(x)}{x^p} dx \le B \int_0^r w(x)dx.$$

B_p Condition

The class B_p has the following remarkable properties (see [Arino and Muckenhoupt, 1990], [M. J. Carro, J. A. Raposo and J. Soria, 2007]):

(i) if $p < q$, then $w \in B_p \Rightarrow w \in B_q$;

(ii) if $w \in B_p$, then there exists $\varepsilon_0 > 0$ such that $w \in B_{p-\varepsilon_0}$.

Let us denote by D the class of all non–negative non–increasing functions on I. Under the symbol $L^{p),\theta}_{dec,w}(I)$ we mean the intersection $L^{p),\theta}_{w}(I) \cap D$.

Lorentz Spaces. Theorem by Arino and Muckenhoupt

The weighted Lorentz space Λ_w^p is defined as a set of functions g on $\mathbf{R}^n$ such that

$$\|g\|_{\Lambda_w^p} = \left[\int_0^\infty [g^*(x)]^p w(x) dx \right]^{1/p} < \infty,$$

where g^* is the decreasing rearrangement of g:

$$g^*(t) = \inf\{\lambda : |\{x \in \mathbf{R}^n : |g(x)| > \lambda\}| \leq t\}.$$

If $w(x) = x^{p/q-1}$, then we have usual Lorentz space denoted by $L^{p,q}$. If $w \equiv$ *const* then it is a classical Lebesgue space.
The next statement gives solution of the one–weight problem for the operator M in weighted Lorentz spaces Λ_w^p defined on $\mathbf{R}^n$ (see [Arino and Muckenhoupt, 1990]):

Theorem B. *Let $0 < p < \infty$ and let w be a non–negative function on $\mathbf{R}_+$. Then the Hardy–Littlewood maximal operator M is bounded in Λ_w^p if and only if $w \in B_p$.*

Let supp $w \subset I$. Together with grand Lebesgue spaces we are interested in the space $\Lambda_w^{p),\theta}$, which is defined as follows:

$$\Lambda_w^{p),\theta} := \Bigg\{ g : \|g\|_{\Lambda_w^{p),\theta}} := \sup_{0<\varepsilon\leq\varepsilon_0} \left(\varepsilon^\theta \int_0^\infty [g^*(x)]^{p-\varepsilon} w(x)dx \right)^{1/(p-\varepsilon)}$$

$$= \sup_{0<\varepsilon\leq\varepsilon_0} \left(\varepsilon^\theta \int_0^1 [g^*(x)]^{p-\varepsilon} w(x)dx \right)^{1/(p-\varepsilon)} < \infty \Bigg\},$$

where $\varepsilon_0 = p - 1$ if $p > 1$; $0 < \varepsilon_0 < p$ if $p \leq 1$.

It is easy to see that when $w = \ const$ on I, then $\Lambda_w^{p),\theta}$ is the generalized grand Lebesgue space $L^{p),\theta}(\Omega)$, where $|\Omega| = 1$.

Boundedness of the Hardy Transform on the Cone of Non–increasing Functions under $B_p(I)$ Condition

Let $0 < p < \infty$. We say that a non–negative integrable function w on I belongs to the class $B_p(I)$ if there is a positive constant $\widetilde{B}$ such that for all $0 < r \leq 1$ the inequality

$$r^p \int_r^1 \frac{w(t)}{t^p} dt \leq \widetilde{B} \int_0^r w(t) dt$$

holds.

Theorem 6. *Let $0 < p < \infty$ and let $\theta > 0$. Suppose that w is a weight on I. Then the inequality*

$$\|Hf\|_{L_w^{p),\theta}(I)} \leq c\|f\|_{L_{dec,w}^{p),\theta}(I)} \tag{9}$$

holds if and only if $w \in B_p(I)$.

Theorem 7. *Let* $0 < p < \infty$. *Suppose that* w *is a non–negative function on* $\mathbf{R}_+$ *such that supp* $w \subset I$ *and* $w \in L(I)$. *Then the operator* M *is bounded in* $\Lambda_w^{p),\theta}$ *if and only if* $w \in B_p(I)$.

The results of this talk were reflected in the papers: [7-12].
Finally we mention that the results of Section 3 were independently derived in the paper by P. Jain and S. Kumari, On grand Lorentz spaces and the maximal operator. *Submitted*.

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References

[1] T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.* **119** (1992) 129–143.

[2] L. Greco, T. Iwaniec and C. Sbordone, Inverting the p-harmonic operator, *Manuscripta Math*. **92**(1997), 249–258.

[3] A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces, *Collect. Math*. **51 (2)** (2000), 131-148.

[4] A. Fiorenza, B. Gupta and P. Jain, The maximal theorem in weighted grand Lebesgue spaces, *Studia Math.* **188(2)**(2008), 123-133.

[5] V. Kokilashvili, Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem. *Proc. A. Razmadze Math. Inst.* **151**(2009), 129-133.

[6] V. Kokilashvili, Boundedness criteria for singular integrals in weighted grand Lebesgue spaces. *J. Math. Sci.* (Springer, New-York) **170** (2010), No. 1, 20–33.

[7] V. Kokilashvili and A. Meskhi, A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces, *Georgian Math. J.* **16** (2009), Number 3, 547–551.

References

[8] V. Kokilashvili and A. Meskhi, Potentials with product kernels in grand Lebesgue spaces. One–weight criteria. *Preprint,* 2011.
[9] V. Kokilashvili and A. Meskhi, Trace inequalities for integral operators with fractional order in grand Lebesgue spaces, *Preprint*, 2011.
[10] A. Meskhi, Criteria for the boundedness of potential operators in grand Lebesgue spaces, *Preprint*, arXiv:1007.1185v1 [math.FA] 7 Jul 2010.
[11] A. Meskhi, Maximal functions, potentials and singular integrals in grand Morrey spaces, *Complex Variables and Elliptic Equations* **56**, Nos. 10–11, OctoberNovember 2011, 1003-1019
[12] A. Meskhi, Weighted criteria for the Hardy transforms under the B_p condition in grand Lebesgue spaces and some applications. *J. Math. Sci., Springer*, **178**(2011), No. 6, 622–636.