Interpolation of function spaces and extension operators

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Tabarz
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Besov spaces of generalised smoothness on $\mathbb{R}^n$

Definition - Besov space on $\mathbb{R}^n$

Let

- $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence,
- $\varphi = (\varphi_j)_{j\in\mathbb{N}_0}$ be a partition of unity,
- $0 < p, q \leq \infty$.

The *Besov space of generalised smoothness* on $\mathbb{R}^n$ is given by

$$B^{\sigma}_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^{\sigma}_{p,q}(\mathbb{R}^n)\| < \infty\}.$$

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$$\|f|B^{\sigma}_{p,q}(\mathbb{R}^n)\| := \left\{\sum_{j=0}^{\infty} \sigma_j^q \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(\mathbb{R}^n)\|^q\right\}^{1/q}$$

Classical smoothness: $\sigma = (s)$, $s \in \mathbb{R}$

$$\|f|B^{(s)}_{p,q}(\mathbb{R}^n)\| := \left\{\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(\mathbb{R}^n)\|^q\right\}^{1/q}$$

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$\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence:

$\sigma = (\sigma_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}^+$ and there are positive constants d_0, d_1 such that

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \; j \in \mathbb{N}_0.$$

Classical smoothness: $\sigma_j = 2^{js}, \quad \sigma = (s), s \in \mathbb{R}.$

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Classical smoothness: $\sigma_j = 2^{js}, \quad \sigma = (s),\ s \in \mathbb{R}.$

Generalised smoothness:

$$\underline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j},$$

where

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}_0.$$

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$$B^{\sigma}_{p,q}(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f | B^{\sigma}_{p,q}(\mathbb{R}^n) \| < \infty \}.$$

Classical smoothness: $\sigma_j = 2^{js}, \quad \sigma = (s),\ s \in \mathbb{R}.$

Generalised smoothness:

For each $\delta > 0$ there are $c_1 = c_1(\delta), c_2 = c_2(\delta) > 0$ such that for all $j, k \in \mathbb{N}_0$,

$$c_1 2^{(\underline{s}(\sigma) - \delta)j} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq c_2 2^{(\overline{s}(\sigma) + \delta)j}.$$

Definition - *gauge function*

Let $\mathbb{H}$ denote the class of all continuous monotone increasing functions $h : (0,\infty) \to (0,\infty)$ such that $h(0^+) = 0$. We refer to $\mathbb{H}$ as the set of all *gauge functions*.

Definition - *h-set*

Let $h \in \mathbb{H}$ and Γ be a non-empty compact set of $\mathbb{R}^n$. We say that Γ is an *h-set* if there exists a finite Radon measure μ such that

$$\operatorname{supp} \mu = \Gamma,$$

$$\mu(B(\gamma,r)) \sim h(r), \quad 0 < r \leq 1,\, \gamma \in \Gamma.$$

Definition - Trace

We consider an h-set $\Gamma \subset \mathbb{R}^n$, an admissible sequence σ and $0 < p,q < \infty$.

1. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define $\mathrm{tr}_\Gamma \varphi = \varphi|_\Gamma$.
2. We suppose that there exists $c > 0$ such that
$$\|\varphi|_\Gamma \,|\, L_p(\Gamma)\| \leq c\|\varphi \,|\, B^\sigma_{p,q}(\mathbb{R}^n)\|, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1}$$
3. By completion $\mathrm{tr}_\Gamma f$ for all $f \in B^\sigma_{p,q}(\mathbb{R}^n)$.

Definition - Besov spaces on h-sets

Consider an h-set $\Gamma \subset \mathbb{R}^n$ satisfying the ball condition. Let σ be an admissible sequence, with $\underline{s}(\sigma) > 0$ and let $0 < p, q \leq \infty$. Then we define

$$\mathbb{B}^{\sigma}_{p,q}(\Gamma) = \mathrm{tr}_{\Gamma}\, B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n) \tag{2}$$

endowed with the quasi-norm

$$\|f \,|\, \mathbb{B}^{\sigma}_{p,q}(\Gamma)\| = \inf \|g \,|\, B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ such that $\mathrm{tr}_{\Gamma} g = f$. If $p = q$ we denote these spaces by $\mathbb{B}^{\sigma}_{p}(\Gamma)$.

h-set	$\mathbb{R}^n$
σ	$\sigma \boldsymbol{h}^{1/p}(n)^{1/p} = (\sigma_j h(2^{-j})^{1/p} 2^{\frac{nj}{p}})_j$
$(s) = (2^{sj})_j$	$(s)\boldsymbol{h}^{1/p}(n)^{1/p} = (2^{sj} h(2^{-j})^{1/p} 2^{\frac{nj}{p}})_j$

Interpolation with function parameter

Theorem - Interpolation of Besov spaces on $\mathbb{R}^n$

Let σ be an admissible sequence and ϕ a convenient function satisfying $\phi(2^j) \sim \sigma_j$, $j \in \mathbb{N}$. Let $0 < p \leq \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < s_0$ and γ be given by

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0 - s_1}}}{\phi\left(t^{\frac{1}{s_0 - s_1}}\right)}, \quad t \in (0, \infty). \tag{3}$$

Then

$$\left(B^{(s_0)}_{p,q_0}(\mathbb{R}^n), B^{(s_1)}_{p,q_1}(\mathbb{R}^n)\right)_{\gamma,q} = B^{\sigma}_{p,q}(\mathbb{R}^n). \tag{4}$$

References: Merucci (1984), complemented by Cobos and Fernandez (1988) and Almeida (2005).

Example - Pointwise multipliers

Theorem - pointwise multipliers

Let σ be an admissible sequence, $0 < p, q \leq \infty$ and ρ satisfy

$$\rho > \max\left\{\bar{s}(\sigma), \frac{n}{p} - \underline{s}(\sigma)\right\}. \tag{5}$$

Then any $g \in \mathcal{S}(\mathbb{R}^n)$ is a multiplier for $B^{\sigma}_{p,q}(\mathbb{R}^n)$, i.e., $f \mapsto gf$ yields a bounded linear mapping from $B^{\sigma}_{p,q}(\mathbb{R}^n)$ into itself and there is a positive constant c such that

$$\|gf|B^{\sigma}_{p,q}(\mathbb{R}^n)\| \leq c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\| \cdot \|f|B^{\sigma}_{p,q}(\mathbb{R}^n)\|, \tag{6}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$ and all $f \in B^{\sigma}_{p,q}(\mathbb{R}^n)$.

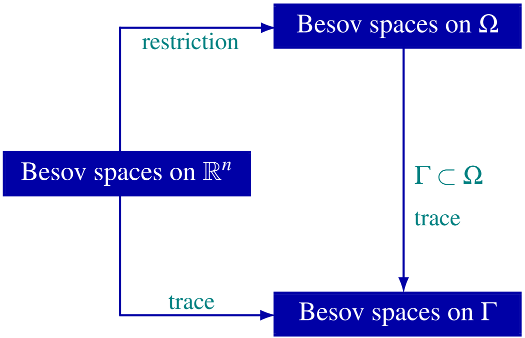
Besov spaces on domains

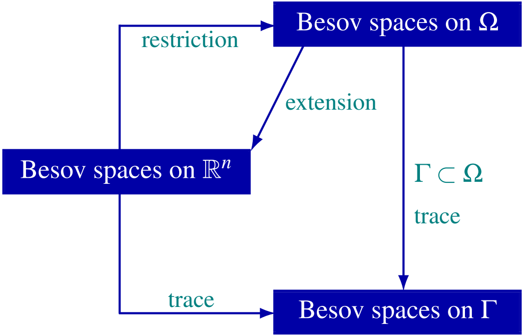
Definition

Let Ω be a domain in $\mathbb{R}^n$, σ be an admissible sequence, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $B^{\sigma}_{p,q}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in B^{\sigma}_{p,q}(\mathbb{R}^n)$ with $g|_{\Omega} = f$. Furthermore,

$$\|f|B^{\sigma}_{p,q}(\Omega)\| := \inf \|g|B^{\sigma}_{p,q}(\mathbb{R}^n)\|, \tag{7}$$

where the infimum is taken over all $g \in B^{\sigma}_{p,q}(\mathbb{R}^n)$ such that its restriction $g|_{\Omega}$ to Ω coincides in $D'(\Omega)$ with f.


Besov spaces on Ω
restriction
Besov spaces on $\mathbb{R}^n$
$\Gamma \subset \Omega$
trace
trace
Besov spaces on Γ


Besov spaces on Ω
restriction
extension
Besov spaces on $\mathbb{R}^n$
Γ ⊂ Ω
trace
trace
Besov spaces on Γ

Theorem - Universal extension operator

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There is a universal extension operator, i.e., there is an extension operator $\mathcal{E}$ such that, for all admissible sequences σ and $0 < p, q \leq \infty$,

$$\mathcal{E} : B^{\sigma}_{p,q}(\Omega) \to B^{\sigma}_{p,q}(\mathbb{R}^n).$$

Proposition - Interpolation of function spaces on domains

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let σ be an admissible sequence, $0 < p \leq \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \leq \bar{s}(\sigma) < s_0$ and γ be given by

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0 - s_1}}}{\phi\left(t^{\frac{1}{s_0 - s_1}}\right)}, \quad t \in (0, \infty), \tag{8}$$

where ϕ is convenient function satisfying $\phi(2^j) \sim \sigma_j$, $j \in \mathbb{N}$. Then

$$\left(B^{(s_0)}_{p,q_0}(\Omega), B^{(s_1)}_{p,q_1}(\Omega)\right)_{\gamma,q} = B^{\sigma}_{p,q}(\Omega).$$

Proposition

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let σ be an admissible sequence and $s_0, s_1 \in \mathbb{R}$ such that

$$\frac{1}{p} + \max\left(0, (n-1)\left(\frac{1}{p} - 1\right)\right) < s_1 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < s_0$$

and γ be as previously. Then

$$\left(\{f \in B^{(s_0)}_{p,q_0}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}, \{f \in B^{(s_1)}_{p,q_1}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}\right)_{\gamma,q}$$

$$= \{f \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}.$$

The Laplacian in Ω

$$-\Delta := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

Theorem - Dirichlet Laplacian in general Besov spaces

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$,

$$1 \le p \le \infty, \quad 1 \le q \le \infty$$

and σ be an admissible sequence such that

$$\underline{s}(\sigma) > \frac{1}{p}.$$

Then $-\Delta$ maps

$$\{g \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega} g = 0\} \quad \text{isomorphically onto} \quad B^{\sigma(-2)}_{p,q}(\Omega).$$

The fractal Dirichlet problem

Theorem

Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$ and Γ be an h-set with $\Gamma \subset \Omega$. Suppose that

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n.$$

Let $g \in \mathbb{H}^\sigma(\Gamma)$, where

$$\sigma := (1)\boldsymbol{h}^{-1/2}(n)^{-1/2} = (2^j h(2^{-j})^{-1/2} 2^{-\frac{nj}{2}})_{j \in \mathbb{N}_0}.$$

Then the Dirichlet problem

$$u \in H^1(\Omega), \quad \Delta u(x) = 0 \quad \text{in} \quad \Omega \setminus \Gamma, \tag{9}$$

$$\mathrm{tr}_{\partial\Omega} u = 0, \quad \mathrm{tr}_\Gamma u = g, \tag{10}$$

has at least one solution.

Wavelet frames for function spaces on domains

Classic Besov spaces on bounded Lipschitz domains

$$f \in B^{(s)}_{p,q}(\Omega) \qquad \leftrightarrow \qquad f = \sum_{j,l} \underbrace{\lambda^j_l(f) 2^{-\frac{jn}{2}}}_{\text{coeficient}} \times \underbrace{\phi^j_l}_{\text{wavelet}}$$

$$\|f|B^{(s)}_{p,q}(\Omega)\| \sim \|\lambda(f)|b^{(s)}_{p,q}(\mathbb{Z}^\Omega)\|$$

References: Triebel

General Besov spaces on bounded Lipschitz domains

$$f \in B^{\sigma}_{p,q}(\Omega) \qquad \leftrightarrow \qquad f = \sum_{j,l} \underbrace{\lambda^j_l(f) 2^{-\frac{jn}{2}}}_{\text{coeficient}} \times \underbrace{\phi^j_l}_{\text{wavelet}}$$

$$\|f|B^{\sigma}_{p,q}(\Omega)\| \sim \|\lambda(f)|b^{\sigma}_{p,q}(\mathbb{Z}^\Omega)\|$$

Definition - $b^{\sigma}_{p,q}(\mathbb{Z}^{\Omega})$

Let Ω be a bounded domain in $\mathbb{R}^n$ and let

$$\mathbb{Z}^{\Omega} = \{x^j_l \in \Omega : j \in \mathbb{N}_0, l = 1, ..., N_j\},$$

with $N_j \sim 2^{jn}$, such that for some $c_1 > 0$,

$$|x^j_l - x^j_{l'}| \geq c_1 2^{-j}, \, j \in \mathbb{N}_0, \; l \neq l'.$$

Let σ be an admissible sequence and $0 < p, q \leq \infty$. Then $b^{\sigma}_{p,q}(\mathbb{Z}^{\Omega})$ is the collection of all sequences

$$\lambda = \{\lambda^j_l \in \mathbb{C} : j \in \mathbb{N}_0, \; l = 1, ..., N_j\}$$

such that

$$\|\lambda | b^{\sigma}_{p,q}(\mathbb{Z}^{\Omega})\| = \left(\sum_{j=0}^{\infty} \sigma_j^q 2^{-\frac{jnq}{p}} \left(\sum_{l=1}^{N_j} |\lambda^j_l|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \tag{11}$$

Definition - wavelet system

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $u \in \mathbb{N}$. Then

$$\Phi = \{\phi_l^j : j \in \mathbb{N}_0, l = 1, \ldots, N_j\} \subset C^u(\Omega)$$

is called a u-wavelet system in $\overline{\Omega}$ if for some $c_2 > 0$ and $c_3 > 0$,

$$\text{supp } \phi_l^j \subset B(x_l^j, c_2 2^{-j}) \cap \overline{\Omega},\ j \in \mathbb{N}_0,\ l = 1, \ldots, N_j,$$

and

$$|D^\alpha \phi_l^j(x)| \leq c_3 2^{\frac{jn}{2} + j|\alpha|},\ j \in \mathbb{N}_0,\ l = 1, \ldots, N_j,\ x \in \Omega,$$

for $\alpha \in \mathbb{N}_0{}^n$ with $0 \leq |\alpha| \leq u$.

Theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. For any $u \in \mathbb{N}$ there is a u-wavelet system $\{\phi_l^j\}$ with the following property. Let σ be an admissible sequence, $0 < p, q < \infty$ and

$$u > \max(\bar{s}(\sigma), n(1/p-1)_+ - \underline{s}(\sigma)).$$

Then $f \in D'(\Omega)$ is an element of $B^\sigma_{p,q}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j 2^{-jn/2} \phi_l^j, \quad \lambda \in b^\sigma_{p,q}(\mathbb{Z}^\Omega), \tag{12}$$

unconditional convergence being in $B^\sigma_{p,q}(\Omega)$. Furthermore

$$\|f|B^\sigma_{p,q}(\Omega)\| \sim \inf \|\lambda|b^\sigma_{p,q}(\mathbb{Z}^\Omega)\|, \tag{13}$$

where the infimum is taken over all representations (12) (equivalent quasi-norms). Any $f \in B^\sigma_{p,q}(\Omega)$ can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j(f) 2^{-jn/2} \phi_l^j, \tag{14}$$

where $\lambda_l^j(\cdot) \in B^\sigma_{p,q}(\Omega)'$ are linear and continuous functionals on $B^\sigma_{p,q}(\Omega)$ and

$$\|f|B^\sigma_{p,q}(\Omega)\| \sim \|\lambda(f)|b^\sigma_{p,q}(\mathbb{Z}^\Omega)\|, \tag{15}$$

(u-wavelet frame).

Thank you!