Sharp constants in norm inequalities in Lorentz spaces

Liviu-Gabriel Marcoci

Technical University of Civil Engineering and
University of Bucharest
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Hardy operator and B_p weights

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We will write $w \in B_p$, if w is a nonnegative locally integrable function in $\mathbb{R}_+$ and

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We denote by $W(t) = \int_0^t w(s)ds$.

Theorem (AM-S)

For $0 < p < \infty$ the following statements are equivalent:
(i) $w \in B_p$;
(ii) $\int_r^\infty \left(\frac{r}{t}\right)^p w(t)dt \le C \int_0^r w,\ r > 0$;
(iii) $\int_0^r \frac{t^{p-1}}{W(t)}dt \le C\frac{r^p}{W(r)},\ r > 0$.
(iv) $\int_0^r \frac{1}{W^{\frac{1}{p}}(t)}dt \le C\frac{r}{W^{\frac{1}{p}}(r)},\ r > 0$.

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$$d(w, p) = \{x : \|x\|_{p,w} := \left(\sum_{n=1}^{\infty} (x_n^*)^p w_n\right)^{\frac{1}{p}} < \infty\}.$$

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$\|\cdot\|_{p,w}$ is a norm if and only if w is a decreasing sequence (see e.g. M. J. Carro, J. A. Raposo and J. Soria, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Mem. Amer. Math. Soc. **187**, 2007).

Also $d(w,p)$ is equivalently normable if and only if

$$\sum_{k=0}^{n} \frac{1}{W_k^{1/p}} \leq C \frac{n+1}{W_n^{1/p}}, n = 0, 1, \dots \tag{1}$$

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The condition (1) characterizes the boundedness of the discrete Hardy operator

$$A_d(x_n) = \frac{1}{n+1} \sum_{k=0}^{n} x_n, \quad n \in \mathbb{N}$$

from $d(w,p)$ to $\ell^p(w)$.

As a consequence of the fact that $\|\cdot\|_{p,w}$ is equivalent to a norm if w satisfies the condition (1), it is easy to see that it is a quasi-norm satisfying the triangle inequality uniformly in the numbers of terms expressed as follows: there exists a constant $C_{p,w} > 0$ such that, for every finite collection $\{x^{(k)}\} \subset d(p,w)$, it yields that

$$\|\sum_{k=1}^{N} x^{(k)}\|_{p,w} \leq C_{p,w} \sum_{k=1}^{N} \|x^{(k)}\|_{p,w}. \tag{2}$$

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It can be proved that also the converse result holds, namely that (2) is equivalent to the fact that $\|\cdot\|_{p,w}$ is normable and, even more, that an alternative equivalent norm is given by means of the following *decomposition norm*:

$$\|x\|_{(p,w)} := \inf\left\{\sum_{k=1}^{N} \|x^{(k)}\|_{p,w} : x = \sum_{k=1}^{N} x^{(k)}\right\}.$$

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If, moreover, w is a decreasing weight sequence then $\|\cdot\|_{(p,w)} = \|\cdot\|_{p,w}$ (see e.g. L. Maligranda 2004).

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We also remark that the best constant $C_{p,w}$ in the inequality

$$\|x\|_{p,w} \leq C_{p,w}\|x\|_{(p,w)}$$

is the same as the optimal one in (2).

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(b) If $0 \leq y \leq x$, then $\|y\|_{(p,w)} \leq \|x\|_{(p,w)}$.
(c) If $0 \leq y^{(k)} \leq x$ and $y^{(k)} \nearrow x$ when $k \to \infty$, then $\|y^{(k)}\|_{(p,w)} \to \|x\|_{(p,w)}$.

The notion of level function was introduced in the early 1950's by I. Halperin and G. G. Lorentz and generalized more recently by G. Sinnamon in a series of papers. Based on the extension given by G. G. Lorentz the optimal constant in the triangle inequality in Lorentz spaces $L^{p,s}(R,\mu)$, where (R,μ) is a totally σ-finite nonatomic measure space and $1 < p < s \leq \infty$ was found by S. Barza, V. Kolyada and J. Soria (2009).

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Level sequences

Let $\varphi = (\varphi_n)$ be a sequence of positive numbers, $\Phi_n = \sum_{k=0}^{n} \varphi_k$, $n \geq 0$ and let $x = (x_n)$ be a sequence of positive numbers such that

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$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} x_i}{\sum_{i=0}^{n} \varphi_i} = 0. \tag{4}$$

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(a) $\left(\frac{x_n^\circ}{\varphi_n}\right)$ is decreasing;

(b) $x \prec x^\circ$;

(c) The set $\{n : x_n^\circ \neq x_n\} = \cup_{k=1}^{\infty} I_k$, where $\{I_k\}$ are finite sets, $I_k = \{n_k, \ldots, n_k + m_k\}$ such that $\sum_{i \in I_k} x_i = \sum_{i \in I_k} x_i^\circ$ and $\frac{x_i^\circ}{\varphi_i} = \lambda_k$, for all $i \in I_k$.

Example of level function

For $f(t) = \chi_{[0,1]}(t)$ the level function with respect to the function $\varphi = t^{-\alpha}$ with $\alpha = 1 - \frac{s'}{p'}$ is

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so that

$$\|f^{\circ}\|_{p,s} = \left(\frac{s'}{p'}\right)^{\frac{1}{s'}}.$$

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If $x = (x_n)_n$, where

$$x_n = \begin{cases} 1 \text{ if } n \leq k, \\ 0 \text{ if } n > k, \end{cases}$$

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$$x_n^\circ = \begin{cases} \frac{k}{\sum_{i=1}^{k} i^{-\alpha}} n^{-\alpha} & \text{if } n \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

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and

$$\|x^\circ\|_{p,s} = k \left(\sum_{n=1}^{k} n^{-\alpha} \right)^{-\frac{1}{s'}} .$$

Muckenhoupt, 1972

Let $p > 1$. A weight function w is said to belong to the $A_p(0,\infty)$ class if

$$\frac{1}{b-a}\int_a^b w(t)dt \leq C\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)dt\right)^{p-1},$$

for some constant C and any $0 < a < b < \infty$. We also denote by

$$\|w\|_{A_p(0,\infty)} = \sup_{0<a<b}\left(\frac{1}{b-a}\int_a^b w(t)dt\right)\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)dt\right)^{1-p}$$

the optimal constant in the above inequality.

Lemma (Barza-Soria, 2010)

Let w be a monotone weight in $A_p(0,\infty)$, for some $p>1$. Then,

$$\|w\|_{A_p(0,\infty)} = \sup_{r>0}\left(\frac{1}{r}\int_0^r w(t)dt\right)\left(\frac{1}{r}\int_0^r w^{1-p'}(t)dt\right)^{p-1}.$$

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Let $p>1$ and w a weight sequence. Let us define

$$C^*_{p,w} = \sup_{m,n\in\mathbb{N}^*} \left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_k\right)\left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_n^{1-p'}\right)^{p-1}$$

Lemma

Let $w = (w_n)$ be a monotone weight sequence and $p > 1$. Then

$$C^*_{p,w} = \sup_{n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{k=1}^{n} w_k \right) \left(\frac{1}{n} \sum_{k=1}^{n} w_k^{1-p'} \right)^{p-1} . \tag{5}$$

Theorem

Let $1 < p < \infty$ and w be an increasing weight. Suppose that $x \in d(p, w)$ is an nonnegative and non-increasing sequence. Let x° be the level sequence of x with respect with the sequence $\varphi = w^{1-p'}$. Then

$$\|x^\circ\|_{p,w} \leq \|x\|_{p,w} \leq C^*_{p,w} \cdot \|x^\circ\|_{p,w}. \tag{6}$$

The constants in inequalities (6) are optimal.

We have the following Hölder inequality for the Lorentz sequence spaces: $x = (x_n) \in d(p, w)$ and $y = (y_n) \in d(p', \widetilde{w})$, $1 < p < \infty$, where $\widetilde{w} = w^{1-p'}$

$$\left| \sum_{n=1}^{\infty} x_n y_n \right| \leq \|x\|_{p,w} \|y\|_{p',\widetilde{w}}. \tag{7}$$

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It is also natural to consider the norm defined in terms of Köthe duality, which is denoted as the *dual norm*

$$\|x\|'_{p,w} = \sup \left\{ \sum_{n=1}^{\infty} x_n y_n : \|y\|_{p',\widetilde{w}} = 1 \right\}.$$

Theorem

Let $1 < p < \infty$*. Assume that* $x \in d(p, w)$ *is a nonnegative and non-increasing sequence and* w *satisfying the condition* (1) *an increasing weight. Set* $\varphi = w^{1-p'}$*. Then*

$$\|x\|'_{p,w} = \|x^\circ\|_{p,w},$$

where x° *is the level sequence of* x *with respect to the sequence* φ*.*

One of the main results is the coincidence of the dual and decomposition norms.

Theorem

Let $1 < p < \infty$ *and* w *satisfying the condition* (1) *an increasing weight. Then, for any sequence* $x \in d(p, w)$*,*

$$\|x\|'_{p,w} = \|x\|_{(p,w)}.$$

Theorem

Let $1 < p < \infty$ *and* w *satisfying the condition* (1) *an increasing weight. Assume that* $x^{(k)} \in d(p,w)$ $(k = 1, 2, \ldots, N)$*. Then*

$$\left\| \sum_{k=1}^{N} x^{(k)} \right\|_{p,w} \leq C^*_{p,w} \sum_{k=1}^{N} \|x^{(k)}\|_{p,w}$$

and the constant is optimal.