On moduli of p-continuity

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p-variation: Definition

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$$v_p(f) := \sup \left(\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p \right)^{1/p} < \infty$$

where the supremum is taken over all sets of points $x_0 < x_1 < ... < x_n = x_0 + 1$.

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For $p = 1$, C. Jordan, 1881.

$1 \leq p < \infty$, N. Wiener, 1924.

The modulus of p-continuity

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$$\omega_{1-1/p}(f;\delta) := \sup \left(\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p \right)^{1/p},$$

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For $1 < p < \infty$, the quantity $\omega_{1-1/p}(f;\delta)$ may tend to 0 as $\delta \to 0$ for non-constant functions. Obviously, all such functions are continuous.

Statement of main problem

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PROBLEM Find sharp reverse relations. Given $f \in V_q$, how much smoothness (measured by $\omega_{1-1/q}(f;\delta)$) must we impose in order to have $f \in V_p$, $1 < p < q < \infty$? Obtain estimates of $\omega_{1-1/p}(f;\delta)$ in terms of $\omega_{1-1/q}(f;\delta)$.

Analogy: the L^p-case

Let $f \in L^p \equiv L^p[0,1]$ $(1 \le p < \infty)$, the L^p-**modulus of continuity** is given by

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We have the following theorem, due to Hardy and Littlewood. Let $1 \leq p < q < \infty$ and $0 < \alpha \leq 1$. If

$$\omega(f;\delta)_p = O(\delta^\alpha)$$

and $\theta \equiv 1/p - 1/q < \alpha$, then $\omega(f;\delta)_q = O(\delta^{\alpha - \theta})$.

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THEOREM (P. Ul'yanov, 1968) Let $1 \leq p < q < \infty$ and ω any modulus of continuity, then

$$H_p^\omega \subset L^q \quad \iff \quad \int_0^1 [t^{-\theta}\omega(t)]^q \frac{dt}{t} < \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

The L^p-case (contd.)

Ul'yanov also obtained the estimate

$$\omega(f;\delta)_q \leq c_{p,q}\left(\int_0^\delta [t^{-\theta}\omega(f;t)_p]^q \frac{dt}{t}\right)^{1/q}, \tag{1}$$

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Andrienko proved that this estimate is sharp: there exists a constant $c_{p,q} > 0$ such that given any modulus of continuity ω and any $\delta \in (0,1]$, there is a function $f \equiv f_\delta \in L^p$ such that $\omega(f_\delta;t)_p \leq \omega(t)$, $t \in [0,1]$ and

$$\omega(f_\delta;\delta)_q \geq c_{p,q} \left(\int_0^\delta [t^{-\theta}\omega(t)]^q \frac{dt}{t} \right)^{1/q} \tag{2}$$

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In general, it is not possible to choose a function f such that (2) holds for all $\delta \in (0,1]$. Moreover, (1) can be improved "in average" (Kolyada, 1988).

Main results

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THEOREM (V. Kolyada & M.L.) Let $1 < p < q < \infty$ and ω be any modulus of q-continuity. Then

$$V_q^\omega \subset V_p \quad \Longleftrightarrow \quad \int_0^1 [t^{-\theta}\omega(t)]^q \frac{dt}{t} < \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

Main results (contd.)

Also, we obtain the estimate

$$\omega_{1-1/p}(f;\delta) \le 4\left(\int_0^\delta [t^{-\theta}\omega_{1-1/q}(f;t)]^q \frac{dt}{t}\right)^{1/q}, \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

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In comparison to the Ul'yanov inequality for L^p-modulus of continuity, our estimate is sharp in the following strong way.

THEOREM (V. Kolyada & M.L.) Assume that $1 < p < q < \infty$ and that ω is any modulus of q-continuity. There is a constant $c_{p,q} > 0$ and a function $f \in V_q$ with $\omega_{1-1/q}(f;t) \leq \omega(t)$, $t \in [0,1]$ such that

$$\omega_{1-1/p}(f;\delta) \geq c_{p,q} \left(\int_0^\delta [t^{-\theta} \omega(t)]^q \frac{dt}{t} \right)^{1/q}$$

for all $\delta \in (0,1]$.

Main results: reformulation

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Let $1 \le p < q < \infty$ and ω any nonnegative and nondecreasing function. Set

$$\rho_{p,q,\omega}(\delta) = \left(\int_0^\delta [t^{-\theta}\omega(t)]^q \frac{dt}{t}\right)^{1/q}, \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

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Let $1 \leq p < q < \infty$ and ω be any modulus of continuity. Then the above given results for L^p can be formulated as

$$0 < c'_{p,q} \leq \inf_{0<\delta\leq 1} \sup_{f\in \overline{H}_p^\omega} \frac{\omega(f;\delta)_q}{\rho_{p,q,\omega}(\delta)} \leq \sup_{f\in \overline{H}_p^\omega} \sup_{0<\delta\leq 1} \frac{\omega(f;\delta)_q}{\rho_{p,q,\omega}(\delta)} \leq c_{p,q}.$$

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$$0 < c_{p,q} \leq \sup_{f \in \overline{V}_q^\omega} \inf_{0<\delta\leq 1} \frac{\omega_{1-1/p}(f;\delta)}{\rho_{p,q,\omega}(\delta)} \leq \sup_{f \in \overline{V}_q^\omega} \sup_{0<\delta\leq 1} \frac{\omega_{1-1/p}(f;\delta)}{\rho_{p,q,\omega}(\delta)} \leq 4.$$

To demonstrate the difference, note that we have the following.

If $F : X \times Y \to \mathbb{R}$, then

$$\sup_{x \in X} \inf_{y \in Y} F(x, y) \leq \inf_{y \in Y} \sup_{x \in X} F(x, y),$$

(exercise).