# Elliptic operators and function spaces on quasi-bounded domains

Leszek Skrzypczak
(joint work with Hans-Gerd Leopold)

Adam Mickiewicz University, Poznań(Poland)
lskrzyp@amu.edu.pl

Tabarz, September 2011

## $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.

# $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact

# $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact
  [2] Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\mathbb{R}^n)$ has purely essential spectrum (the discrete spectrum is empty)

## $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact
  [2] Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\mathbb{R}^n)$ has purely essential spectrum (the discrete spectrum is empty)
- Elliptic operators and function spaces on bounded domain $\Omega \subset \mathbb{R}^n$.

.

## $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact
  [2] Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\mathbb{R}^n)$ has purely essential spectrum (the discrete spectrum is empty)
- Elliptic operators and function spaces on bounded domain $\Omega \subset \mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\Omega) \hookrightarrow W^{s_2}_{p_2}(\Omega)$ is compact if $s_1 - s_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$

.

# $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact
  [2] Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\mathbb{R}^n)$ has purely essential spectrum (the discrete spectrum is empty)
- Elliptic operators and function spaces on bounded domain $\Omega \subset \mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\Omega) \hookrightarrow W^{s_2}_{p_2}(\Omega)$ is compact if $s_1 - s_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$
  [2] Dirichlet Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\Omega)$ has compact resolvent. In consequence it has purely discrete spectrum (the essential spectrum is empty).

# $\mathbb{R}^n$ versus bounded domains $\Omega \subset \mathbb{R}^n$

- Elliptic operators and function spaces on $\mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2}_{p_2}(\mathbb{R}^n)$ are never compact
  [2] Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\mathbb{R}^n)$ has purely essential spectrum (the discrete spectrum is empty)
- Elliptic operators and function spaces on bounded domain $\Omega \subset \mathbb{R}^n$.
  [1] The Sobolev embeddings $W^{s_1}_{p_1}(\Omega) \hookrightarrow W^{s_2}_{p_2}(\Omega)$ is compact if $s_1 - s_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$
  [2] Dirichlet Laplace operator $-\Delta$ with domain $D(-\Delta) = W^2_2(\Omega)$ has compact resolvent. In consequence it has purely discrete spectrum (the essential spectrum is empty).
- Point of interest: Unbounded domains such that the properties of Dirichlet Laplacian and Sobolev embeddings of spaces defined on these domains are similar to that one on bounded domains.

# Function spaces - definitions

1. Besov spaces on $\mathbb{R}^n$, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in S' : \|f|B^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{sqj}\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_p^q\Big)^{1/q} < \infty\}.$$

# Function spaces - definitions

1. Besov spaces on $\mathbb{R}^n$, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in S' : \|f|B^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{sqj} \|\mathcal{F}^{-1}\varphi_j \mathcal{F}f\|_p^q\Big)^{1/q} < \infty\}.$$

2. Besov spaces on domains, $\Omega \subset \mathbb{R}^n$ - an open set $\Omega \neq \mathbb{R}^n$, $1 < p < \infty$ ,$0 < q \leq \infty$, $s \in \mathbb{R}$,

$$\bar{B}^s_{p,q}(\Omega) = \begin{cases} \{f : f = g|_\Omega,\ g \in B^s_{p,q}(\mathbb{R}^n)\}, & \text{if} \quad s \leq 0, \\ \{f : f = g|_\Omega,\ g \in B^s_{p,q}(\mathbb{R}^n),\ \operatorname{supp} g \subset \overline{\Omega}\} & \text{if} \quad s > 0, \end{cases}$$

$$\|f|\bar{B}^s_{p,q}\| = \inf \|g|B^s_{p,q}\| \ .$$

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains
  [2] Lipschitz domains

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains
  [2] Lipschitz domains
  [3] Uniformly E-porous domains

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains
  [2] Lipschitz domains
  [3] Uniformly E-porous domains
  e.g. $\Omega$ is uniformly E-porous if:
  [a] $\Omega$ is E-thick i.e. one can find for any interior cube $Q^i \subset \Omega$ with

$$\ell(Q^i) \sim \operatorname{dist}(Q^i, \partial\Omega) \sim 2^{-j}, \quad j \geq j_0 \in \mathbb{N}$$

a complementing exterior cube $Q^e \subset \mathbb{R}^n \setminus \Omega$ with

$$\ell(Q^e) \sim \operatorname{dist}(Q^e, \partial\Omega) \sim \operatorname{dist}(Q^e, Q^i) \sim 2^{-j}, \; j \geq j_0 \in \mathbb{N}.$$

$Q^i$, $Q^e$ denote cubes with sides parallel to the axes of coordinates.

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains
  [2] Lipschitz domains
  [3] Uniformly E-porous domains
  e.g. $\Omega$ is uniformly E-porous if:
  [a] $\Omega$ is E-thick i.e. one can find for any interior cube $Q^i \subset \Omega$ with

$$\ell(Q^i) \sim \mathrm{dist}\,(Q^i, \partial\Omega) \sim 2^{-j}, \quad j \geq j_0 \in \mathbb{N}$$

  a complementing exterior cube $Q^e \subset \mathbb{R}^n \setminus \Omega$ with

$$\ell(Q^e) \sim \mathrm{dist}\,(Q^e, \partial\Omega) \sim \mathrm{dist}\,(Q^e, Q^i) \sim 2^{-j}, \, j \geq j_0 \in \mathbb{N}.$$

  $Q^i$, $Q^e$ denote cubes with sides parallel to the axes of coordinates.

# Domains in $\mathbb{R}^n$

- Domain $\Omega$ in $\mathbb{R}^n$ is an nonempty open subset of set $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.
- We will need some assumption concerning regularity of the boundaries of the domains. We choose domains One can think about:
  [1] $C^\infty$ domains
  [2] Lipschitz domains
  [3] Uniformly E-porous domains
  e.g. $\Omega$ is uniformly E-porous if:
  [a] $\Omega$ is E-thick i.e. one can find for any interior cube $Q^i \subset \Omega$ with

$$\ell(Q^i) \sim \operatorname{dist}(Q^i, \partial\Omega) \sim 2^{-j}, \quad j \geq j_0 \in \mathbb{N}$$

a complementing exterior cube $Q^e \subset \mathbb{R}^n \setminus \Omega$ with

$$\ell(Q^e) \sim \operatorname{dist}(Q^e, \partial\Omega) \sim \operatorname{dist}(Q^e, Q^i) \sim 2^{-j}, \; j \geq j_0 \in \mathbb{N}.$$

$Q^i$, $Q^e$ denote cubes with sides parallel to the axes of coordinates.
[b] $\partial\Omega$ is a $d$-set with $n-1 \leq d < n$.

# Unbounded domains in $\mathbb{R}^n$

- An unbounded domain $\Omega$ in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega,|x|\to\infty} \operatorname{dist}(x,\partial\Omega) = 0 \quad .$$

# Unbounded domains in $\mathbb{R}^n$

- An unbounded domain $\Omega$ in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega, |x|\to\infty} \mathrm{dist}\,(x, \partial\Omega) \;=\; 0 \quad .$$

- Examples: Let $\alpha > 0$. The open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$

$$\begin{aligned} \omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : \ |y| < x^{-\alpha}, x > 1\} \qquad \text{and} \\ \Omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : \ |y| < |x|^{-\alpha}\} \quad \text{are quasi-bounded.} \end{aligned}$$

# Unbounded domains in $\mathbb{R}^n$

- An unbounded domain $\Omega$ in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega, |x|\to\infty} \operatorname{dist}(x, \partial\Omega) = 0 \quad .$$

- Examples: Let $\alpha > 0$. The open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$

$$\begin{aligned} \omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : \ |y| < x^{-\alpha}, x > 1\} \qquad \text{and} \\ \Omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : \ |y| < |x|^{-\alpha}\} \quad \text{are quasi-bounded.} \end{aligned}$$

$|\Omega_\alpha| = \infty$ but $|\omega_\alpha| < \infty$ if $\alpha > 0$.

# Unbounded domains in $\mathbb{R}^n$

- An unbounded domain $\Omega$ in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega, |x|\to\infty} \operatorname{dist}(x, \partial\Omega) = 0 \quad .$$

- Examples: Let $\alpha > 0$. The open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$

$$\begin{aligned} \omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : |y| < x^{-\alpha}, x > 1\} \qquad \text{and} \\ \Omega_\alpha &= \{(x,y) \in \mathbb{R}^2 : |y| < |x|^{-\alpha}\} \quad \text{are quasi-bounded.} \end{aligned}$$

  $|\Omega_\alpha| = \infty$ but $|\omega_\alpha| < \infty$ if $\alpha > 0$.
- An unbounded domain is not quasi-bounded if, and only if, it contains infinitely many pairwise disjoint congruent balls (cubes).

# Domains in $\mathbb{R}^n$-remarks

- If $\Omega$ in $\mathbb{R}^n$ is uniformly E- porous domain in $\mathbb{R}^n$ then one can proof the wavelet characterization of the Besov (and Triebel-Lizorkin) spaces (Triebel 2008). In consequence we can reduce the investigation of properties of Sobolev embeddings to investigation of embeddings of some sequence spaces.

# Domains in $\mathbb{R}^n$-remarks

- If $\Omega$ in $\mathbb{R}^n$ is uniformly E- porous domain in $\mathbb{R}^n$ then one can proof the wavelet characterization of the Besov (and Triebel-Lizorkin) spaces (Triebel 2008). In consequence we can reduce the investigation of properties of Sobolev embeddings to investigation of embeddings of some sequence spaces.
- Theorem
  If $\Omega$ is not quasi-bounded then the embedding

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{1}$$

  is never compact.

# Domains in $\mathbb{R}^n$-remarks

- If $\Omega$ in $\mathbb{R}^n$ is uniformly E- porous domain in $\mathbb{R}^n$ then one can proof the wavelet characterization of the Besov (and Triebel-Lizorkin) spaces (Triebel 2008). In consequence we can reduce the investigation of properties of Sobolev embeddings to investigation of embeddings of some sequence spaces.
- Theorem
  If $\Omega$ is not quasi-bounded then the embedding

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{1}$$

  is never compact.
- Of interest are quasi-bounded domains

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

# Quantitative characterizations of compactness

Let $T \in L(X,Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \dots$$

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \dots$$

$$T : X \to Y \quad \text{is compact} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} e_k(T) = 0\,.$$

# Quantitative characterizations of compactness

Let $T \in L(X,Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \dots$$
$$T : X \to Y \quad \text{is compact} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} e_k(T) = 0\,.$$

2. Let $T : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(T)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus.

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \dots$$
$$T : X \to Y \quad \text{is compact} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} e_k(T) = 0\,.$$

2. Let $T : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(T)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus.
   Carl's inequality gives

$$|\mu_k(T)| \leq \sqrt{2}\, e_k(T) \quad \text{for all} \quad k \in \mathbb{N}$$

.

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X)\text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \dots$$
$$T : X \to Y \quad \text{is compact} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} e_k(T) = 0\,.$$

2. Let $T : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(T)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus.
   Carl's inequality gives

$$|\mu_k(T)| \leq \sqrt{2}\, e_k(T) \quad \text{for all} \quad k \in \mathbb{N}$$

.

# Quantitative characterizations of compactness

Let $T \in L(X, Y)$ and $B_X$ be the closed unit ball in Banach space $X$.

1. Entropy numbers. The $k$-th entropy number of $T$, $k \in \mathbb{N}$, is

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$

$$\|T\| = e_1(T) \geq e_2(T) \geq e_3(T) \geq \ldots$$

$$T : X \to Y \quad \text{is compact} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} e_k(T) = 0\,.$$

2. Let $T : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(T)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus.
   Carl's inequality gives

$$|\mu_k(T)| \leq \sqrt{2}\, e_k(T) \quad \text{for all} \quad k \in \mathbb{N}$$

.

3. Asymptotic behaviour of eigenvalues of Dirichlet Laplacian on domains. If $-\Delta$ has compact resolvent then $\lambda_k$ the eigenvalues of $-\Delta$ are related to eigenvalues of $B = (-\Delta)^{-1}$ by $\lambda_k = \mu_k^{-1}(B)$.

# Entropy numbers of Sobolev embedding on domain with finite measure

**Theorem**

*Let $\Omega$ be a uniformly E-porous domain in $\mathbb{R}^n$ with finite Lebesgue measure.*

# Entropy numbers of Sobolev embedding on domain with finite measure

**Theorem**

*Let $\Omega$ be a uniformly E-porous domain in $\mathbb{R}^n$ with finite Lebesgue measure.*

$$\text{The embedding} \quad \bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{2}$$

$$\text{is compact if, and only if,} \quad s_1 - s_2 - \left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ > 0\,. \tag{3}$$

# Entropy numbers of Sobolev embedding on domain with finite measure

### Theorem

*Let $\Omega$ be a uniformly E-porous domain in $\mathbb{R}^n$ with finite Lebesgue measure.*

$$\text{The embedding} \quad \bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{2}$$

$$\text{is compact if, and only if,} \quad s_1 - s_2 - \left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ > 0\,. \tag{3}$$

$$\text{Moreover} \qquad e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{\frac{s_1-s_2}{n}} \tag{4}$$

# Entropy numbers of Sobolev embedding on domain with finite measure

**Theorem**

*Let $\Omega$ be a uniformly E-porous domain in $\mathbb{R}^n$ with finite Lebesgue measure.*

$$\text{The embedding} \quad \bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{2}$$

$$\text{is compact if, and only if,} \quad s_1 - s_2 - \Big(\frac{n}{p_1} - \frac{n}{p_2}\Big)_+ > 0\,. \tag{3}$$

$$\text{Moreover} \qquad e_k\big(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\big) \sim k^{\frac{s_1-s_2}{n}} \tag{4}$$

The interesting case are quasi-bounded domains with infinite Lebesgue measure.

# Box packing number of an open set

- $\Omega$ a nonempty open subset of $\mathbb{R}^n$.

# Box packing number of an open set

- $\Omega$ a nonempty open subset of $\mathbb{R}^n$. Let

$$b_j(\Omega) = \sup \Big\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega\,, \quad Q_{j,m_\ell} \text{ being pairwise disjoint} \\ \text{dyadic cubes of side-length } 2^{-j} \Big\}, \quad j = 0, 1, \dots$$

# Box packing number of an open set

- $\Omega$ a nonempty open subset of $\mathbb{R}^n$. Let

$$b_j(\Omega) = \sup\Big\{k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega\,, \quad Q_{j,m_\ell} \text{ being pairwise disjoint dyadic cubes of side-length } 2^{-j}\Big\}, \quad j = 0, 1, \ldots$$

We have $b_{j_o}(\Omega)2^{-j_o n} \le b_j(\Omega)2^{-jn}$ if $j > j_o > j_o(\Omega)$.

# Box packing number of an open set

- $\Omega$ a nonempty open subset of $\mathbb{R}^n$. Let

$$b_j(\Omega) = \sup \Big\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega\,, \quad Q_{j,m_\ell} \text{ being pairwise disjoint dyadic cubes of side-length } 2^{-j} \Big\}, \quad j = 0, 1, \dots$$

We have $b_{j_o}(\Omega) 2^{-j_o n} \le b_j(\Omega) 2^{-jn}$ if $j > j_o > j_o(\Omega)$.

- A box packing number $b(\Omega)$:

$$b(\Omega) = \sup \Big\{ t \in \mathbb{R}_+ : \limsup_{j\to\infty} b_j(\Omega) 2^{-jt} = \infty \Big\}. \tag{5}$$

# Box packing number of an open set

- $\Omega$ a nonempty open subset of $\mathbb{R}^n$. Let

$$b_j(\Omega) = \sup \Big\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega \,, \quad Q_{j,m_\ell} \text{ being pairwise disjoint dyadic cubes of side-length } 2^{-j} \Big\}, \quad j = 0, 1, \ldots$$

  We have $b_{j_o}(\Omega) 2^{-j_o n} \le b_j(\Omega) 2^{-jn}$ if $j > j_o > j_o(\Omega)$.

- A box packing number $b(\Omega)$:

$$b(\Omega) = \sup \Big\{ t \in \mathbb{R}_+ : \limsup_{j\to\infty} b_j(\Omega) 2^{-jt} = \infty \Big\}. \tag{5}$$

- For any nonempty open set $\Omega \subset \mathbb{R}^n$ we have $n \le b(\Omega) \le \infty$.

# Box packing number of an open set - examples

- If $\Omega$ is of finite measure, then $b(\Omega) = n$ since

$$b_j(\Omega)2^{-jn} \leq |\Omega|.$$

# Box packing number of an open set - examples

- If $\Omega$ is of finite measure, then $b(\Omega) = n$ since

$$b_j(\Omega)2^{-jn} \leq |\Omega|.$$

- If $\Omega$ is unbounded and not quasi-bounded, then $b(\Omega) = \infty$,

# Box packing number of an open set - examples

- If $\Omega$ is of finite measure, then $b(\Omega) = n$ since

$$b_j(\Omega)2^{-jn} \leq |\Omega|.$$

- If $\Omega$ is unbounded and not quasi-bounded, then $b(\Omega) = \infty$,
- But there are quasi-bounded domains such that $b(\Omega) = \infty$.

# Box packing number of an open set - examples

- If $\Omega$ is of finite measure, then $b(\Omega) = n$ since

$$b_j(\Omega)2^{-jn} \le |\Omega|.$$

- If $\Omega$ is unbounded and not quasi-bounded, then $b(\Omega) = \infty$,
- But there are quasi-bounded domains such that $b(\Omega) = \infty$.

**Examples**

Let $\alpha > 0$, $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$ be as above. Then

$$b(\omega_\alpha) = \begin{cases} \frac{1}{\alpha} + 1 & \text{if } 0 < \alpha < 1, \\ 2 & \text{if } \alpha \ge 1, \end{cases} \qquad b(\Omega_\alpha) = \begin{cases} \frac{1}{\alpha} + 1 & \text{if } 0 < \alpha < 1, \\ \alpha + 1 & \text{if } \alpha \ge 1. \end{cases}$$

# Quasi-bounded domains - compactness of embeddings

Theorem

(i) *Let* $b(\Omega) < \infty$. *The embedding*

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{6}$$

*is compact if*

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} = b(\Omega)\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big)_+ . \tag{7}$$

# Quasi-bounded domains - compactness of embeddings

Theorem

(i) *Let* $b(\Omega) < \infty$. *The embedding*

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{6}$$

*is compact if*

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} = b(\Omega)\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big)_+ . \tag{7}$$

*If the embedding (6) is compact and* $\frac{1}{p^*} = 0$, *then* $\delta > 0$.
*If the embedding (6) is compact and* $\frac{1}{p^*} > 0$, *then* $\delta \geq \frac{b(\Omega)}{p^*}$.

# Quasi-bounded domains - compactness of embeddings

## Theorem

(i) *Let* $b(\Omega) < \infty$. *The embedding*

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{6}$$

*is compact if*

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} = b(\Omega)\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big)_+ . \tag{7}$$

*If the embedding (6) is compact and* $\frac{1}{p^*} = 0$, *then* $\delta > 0$.
*If the embedding (6) is compact and* $\frac{1}{p^*} > 0$, *then* $\delta \geq \frac{b(\Omega)}{p^*}$.
(ii) *If* $b(\Omega) = \infty$, *then the embedding (6) is compact if, and only if,* $p_1 \leq p_2$ *and*

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 . \tag{8}$$

# Quasi-bounded domains - entropy numbers of embeddings

**Theorem**

*Let* $s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{b(\Omega)}{p^*}$ *and* $b(\Omega) < \infty$. *If*

$$0 < \liminf_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty\,, \tag{9}$$

*then*

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma} \tag{10}$$

*with*

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

# Quasi-bounded domains - entropy numbers of embeddings

**Theorem**

*Let* $s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{b(\Omega)}{p^*}$ *and* $b(\Omega) < \infty$. *If*

$$0 < \liminf_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty \,, \tag{9}$$

$$\textit{then} \qquad e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma} \tag{10}$$

$$\textit{with} \qquad \gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

**Corollary**

*Let* $\Omega$ *be of finite Lebesgue measure. If the embedding is compact, then* $\gamma = \frac{s_1 - s_2}{n}$.

# Quasi-bounded domains - inverse entropy problem

What is the possible asymptotic behaviour of entropy numbers of the compact embedding of the function spaces defined on domains?

# Quasi-bounded domains - inverse entropy problem

What is the possible asymptotic behaviour of entropy numbers of the compact embedding of the function spaces defined on domains?

## Theorem

*Let $s_1, s_2 \in \mathbb{R}$, $1 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. We assume that $\frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$.*
*For positive real $\gamma$, such that $\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$, there exists a quasi-bounded domain $\Omega$ in $\mathbb{R}^n$ such that*

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma}, \quad k \in \mathbb{N}. \tag{11}$$

*If (11) holds for some quasi-bounded domain $\Omega$ in $\mathbb{R}^n$ and $b(\Omega) < \infty$, then $\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$.*

# Elliptic operators on quasi-bounded domains

$$\text{Let} \qquad A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha$$

be a formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \le 2m$. We assume that $A$ is a positive self-adjoint operator in $L_2(\Omega)$. Then $A = A(x,D)$ is an operator with discrete spectrum $\sigma(A)$ of eigenvalues having no finite accumulation point.

# Elliptic operators on quasi-bounded domains

$$\text{Let} \qquad A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha$$

be a formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \le 2m$. We assume that $A$ is a positive self-adjoint operator in $L_2(\Omega)$. Then $A = A(x,D)$ is an operator with discrete spectrum $\sigma(A)$ of eigenvalues having no finite accumulation point.

Theorem

*Let $\Omega$ be quasi-bounded domain in $\mathbb{R}^n$, such that $b(\Omega) < \infty$ and (9) holds. Let $\lambda_1, \lambda_2, \dots$ be eigenvalues of $A$ ordered by their magnitude and counted according to their multiplicities. Then*

$$\lambda_k \sim k^{\frac{2m}{b(\Omega)}}, \qquad k \in \mathbb{N}.$$

## Examples

Let $\alpha > 0$ and $\alpha \neq 1$. For the open set $\Omega_\alpha \subset \mathbb{R}^2$ we have the following formula for the eigenvalues of the Dirichlet Laplacian

$$\lambda_k(-\Delta) \sim \begin{cases} k^{\frac{2\alpha}{1+\alpha}} & \text{if} \quad 0 < \alpha < 1\,, \\ k^{\frac{2}{1+\alpha}} & \text{if} \quad \alpha > 1\,. \end{cases} \tag{12}$$

## Examples

Let $\alpha > 0$ and $\alpha \neq 1$. For the open set $\Omega_\alpha \subset \mathbb{R}^2$ we have the following formula for the eigenvalues of the Dirichlet Laplacian

$$\lambda_k(-\Delta) \sim \begin{cases} k^{\frac{2\alpha}{1+\alpha}} & \text{if} \quad 0 < \alpha < 1\,, \\ k^{\frac{2}{1+\alpha}} & \text{if} \quad \alpha > 1\,. \end{cases} \tag{12}$$

The assumption (9) is sufficient but not necessary to get the estimates of corresponding entropy numbers. For the domain $\Omega_\alpha$ with $\alpha = 1$ one gets

$$e_k\Big(\bar{B}^{s_1}_{p_1,q_1}(\Omega_1) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega_1)\Big) \sim k^{-\frac{s_1-s_2}{2}} (\log k)^{\frac{s_1-s_2}{2} - (\frac{1}{p_1} - \frac{1}{p_2})},$$

$$\text{and} \qquad \lambda_k(-\Delta) \sim k \log k\,.$$