Remarks on the regularity of elliptic systems.

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FSDONA September 2011

$$A(u) + f(u) = 0$$

$$u\colon \Omega \longrightarrow \mathbb{R}^M , \qquad \Omega \subset \mathbb{R}^N .$$

$$(A(u))^k + f^k(u) = 0, \quad k \ = \ 1, \ldots , \ M .$$

$$\big(A(u)\big)^k \ = \ - \sum_{\alpha=1}^{N} \ D_\alpha A_\alpha^k \big(x, u, D(u)\big) \ ,$$

In the linear case we have

$$A_\alpha^k(x, \eta, \zeta) = \sum_{\beta\, l} a_{\beta\,\alpha}^{lk} D_\beta u^l .$$

The p-Laplacian would read

$$A_\alpha^k(x, \eta, \zeta) = |Du|^{p-2} D_\alpha u^k .$$

We are interested in structure conditions on elliptic systems which still yield the related results known for elliptic equations.

In particular we will consider the following conditions on (A) :
(A, i)

$$\sum_k \sum_\alpha A^k_\alpha(x,\eta,\zeta)\ \zeta^k_\alpha \ \geq\ \lambda\ |\zeta|^p\ ;$$

(A, ii)

$$|A^k_\alpha(x,\eta,\zeta)|\ \leq\ C|\zeta|^{p-1};$$

(A, iii)

$$\sum_\alpha \left(\sum_k A^k_\alpha(x,\eta,\zeta)\ \mu^k \right) \left(\sum_k \mu^k\ \zeta^k_\alpha \right)\ \geq\ 0\ .$$

Condition (A, i) is the "usual" ellipticity condition.
Then (A, ii) implies that we can work in the Sobolev spaces $W^{1,p}$.
The structure condition (A, iii) is satisfied by systems in "strict diagonal form" such as the p-Laplacian for instance.
It allows us to deal with the natural growth condition on f given by
(P)

$$|f(x,\eta,\zeta)| \leq \Lambda|\zeta|^r\ ,$$

with $r = p$.

De Giorgi's Counter example:

Unbounded solutions to linear systems with L^∞-coefficients indicate that everywhere C^α-regularity of weak solutions cannot hold if the system is to "far away" from diagonal form.

This can be made somewhat more precise:

For systems of the form

$$\Delta u^k + \sum_{\alpha\beta l} D_\alpha b^{l\,k}_{\beta\alpha}(x) D_\beta u^l = 0\,.$$

with $N = M \geq 3$, there are unbounded solutions for coefficients $b^{l\,k}_{\beta\alpha}$ with

$$|b^{l\,k}_{\beta\alpha}| \leq \frac{4}{\sqrt{N - \frac{7}{2}}}\,.$$

Regularity results for $p = 2$.

(I)
Weak solutions are in a somewhat better Sobolev Space

(II)
Weak solutions are regular (Hoelder continuous) in an open dense set with estimates on the Hausdorff measure of the sets of singularities.

Those statements are true under the assumption (A,i), (A,ii) and
(P) with

$$2\Lambda\mathcal{M} < \lambda$$

where $\mathcal{M} = \| u \|_\infty$.

(III)
Weak solutions are regular everywhere if the system is in addition in strict diagonal form.

The condition (P) is not optimal. Counter examples exist for $\Lambda\mathcal{M} = \lambda$.
In 1976 the regularity with systems in diagonal form was shown by Wiegner in 1976.
(Hildebrandt-Widman, Caffarelli, with different methods.)

What about the other types of regularity results for

$$\Lambda\mathcal{M} < \lambda\,?$$

(Sometimes referred to as $\Lambda\mathcal{M} < 1$ - condition)

$p = 2$

With the structure condition (A, iii) we had been able to give a positve answer with an “angle condition” on the perturbation f which includes the

$\mathcal{M}\lambda < 1$ - condition.

We also could regain Wiegners result, using

“Giaquinta’s” direct approach to regularity.

$p \neq 2$

In a joint work with Marina Borovikova we considered the related “ degenerated “ cases $p \neq 2$, and showed:

$$(A, i), (A, ii), (A, iii) \text{ and } (P) \Rightarrow (I) \text{ and } (II)\,.$$

But we could not verify (III).

Caffarelli's proof of (III).

It rests on the fact that $M^2 - |v|^2$ is a supersolution of the related elliptic equation, where $v = u + \xi$ for constant vectors ξ, with norm small enough. That implies it satisfies the Harnack inequality

$$R^{n/r} \| u \|_{L^r(B(2R)} < C \inf_{B(R)} u\,.$$

For nonnegative functions ϕ in $W^{1,2}_{\circ}(\Omega)$ we have:

$$\frac{1}{2} D(M^2 - |v|^2) D\phi = \sum_{\alpha} \sum_{i} - D_\alpha v^i v^i D_\alpha \phi$$

$$= -\sum_{i} \sum_{\alpha} D_\alpha v^i D_\alpha (v^i \phi) + (\sum_{i} \sum_{\alpha} D_\alpha v^i D_\alpha v^i)\phi$$

Since $Dv = Du$ we get

$$\frac{1}{2} \int D(M^2 - |v|^2) D\phi dx$$

$$= \int -(f(u), v)\phi + |D(u)|^2 \phi dx$$

$$\geq \int \gamma |D(u)|^2 \phi dx \geq 0$$

by assumption.

Now I would like to read this estimates in the opposite direction to prove the Harnack inequality for $(M^2 - |v|^2)$ directly. Using Moser's iteration scheme, we need:

$$\phi(s_1, r_1) \leq \phi(s_2, r_2)$$

with

$$\phi(s, r) + (\int_{B(r)} u^s dx)^{1/s} .$$

Still for $\phi \geq 0$ we have

$$\gamma \int |D(u)|^2 \phi dx \leq \int -\sum_i \sum_\alpha D_\alpha v^i v^i D_\alpha \phi dx$$

For $\phi = (M^2 - |v|^2)^\beta \eta^2$ with $Dv = Du$ we have

$$D_\alpha \phi = (M^2 - |v|^2)^\beta 2\eta D_\alpha \eta$$

$$-\eta^2 2\beta (M^2 - |v|^2)^{\beta - 1} \sum_i v^i D_\alpha u^i$$

and get

$$0 < \gamma \int (|D(u)|^2 (M^2 - |v|^2)^\beta \eta^2) dx$$

$$\leq \beta \int \sum_\alpha (\sum_i D_\alpha u^i v^i \sum_i v^i D_\alpha u^i)(M^2 - |v|^2)^{\beta-1} \eta^2 dx$$

$$- \int \sum_\alpha \sum_i D_\alpha u^i v^i (M^2 - |v|^2)^\beta 2\eta D_\alpha \eta dx$$

For $\beta < 0$ we estimate

$$|\beta| \int \sum_\alpha (\sum_i D_\alpha u^i v^i \sum_i D_\alpha u^i v^i)(M^2 - |v|^2)^{\beta-1} \eta^2 dx$$

$$\leq - \int \sum_\alpha (\sum_i D_\alpha u^i v^i)(M^2 - |v|^2)^\beta 2\eta D_\alpha \eta dx$$

$$\leq (1/2)|\beta| \int \sum_\alpha (\sum_i D_\alpha u^i v^i)^2 (M^2 - v^2)^{\beta-1} \eta^2 dx$$

$$+ \frac{2}{|\beta|} \int \int \sum_\alpha (M^2 - |v|^2)^{\beta+1} (D^\alpha \eta)^2 dx$$

and get

$$\int (M^2 - |v|^2)^{\beta-1} \eta^2 \sum_\alpha (\sum_i D_\alpha u^i v^i \sum_i D_\alpha u^i v^i) dx$$

$$\leq C|\beta| \int (M^2 - |v|^2)^{\beta+1}) \sum_\alpha (D_\alpha \eta)^2 dx$$

Now to verify the iteration inequality we consider

$$\int \eta^2|D((M^2-|v|^2)^\kappa)|^2dx$$

$$=\int \eta^2((\kappa((M^2-|v|^2)^{\kappa-1})\sum_\alpha(D_\alpha\sum_i -(v^i)^2))^2dx$$

$$\leq c\int \eta^2((M^2-|v|^2)^{2\kappa-2})\sum_\alpha(\sum_i D^\alpha u^i v^i)^2dx$$

$$\leq C(\kappa,N)\int\sum_\alpha(M^2-|v|^2)^{2\kappa}(D^\alpha\eta)^2dx$$

Which gives the pivotal inequality to apply Moser"s technic.

Replacing the linear term $D_\alpha u^i$ with
$A^i_\alpha = A^i_\alpha(x,u,Du)$ we would like to get the analogu-ous estimates.

$$\int(M^2-|v|^2)^{\beta-1}\eta^p(\sum_\alpha\sum_i A^i_\alpha v^i\sum_i D_\alpha u^i v^i)^p dx$$

$$\leq C|\beta|\int(M^2-|v|^2)^{p+\beta-1})\sum_\alpha(D_\alpha\eta)^p dx$$

and

$$\int \eta^p |(D((M^2 - |v|^2)^\kappa)|^p dx$$

$$\leq c \int \eta^2((M^2 - |v|^2)^{p\kappa - p}) \sum_\alpha (\sum_i D_\alpha u^i v^i)^p dx$$

$$\leq C(\kappa, N) \int \sum_\alpha (M^2 - |v|^2)^{p\kappa} (D^\alpha \eta)^p dx$$

The first one is certainly satisfied if we have

(S,i) $\sum_\alpha \sum_i A^i_\alpha \eta^i ab$

$$\leq c_1 (\sum_\alpha \sum_i A^i_\alpha \eta^i (\sum_i \eta^i \zeta^i_\alpha) a^{p/(p-1)} + c_2 b^p$$

and for the second

(S,ii) $\sum_\alpha (\sum_i \zeta^i_\alpha \eta^i)^p \leq c \sum_\alpha (\sum_i A^i_\alpha(\zeta) \eta^i)(\sum_i \zeta^i_\alpha \eta^i)$

For $p = 2$ those inequalities can be easily verified for systems in diagonal form with the Laplacian as the elliptic operator.

We can restate the regularity result (for $p = 2$)

$$(A, i), (A, ii), (A, iii), (S, i), (S, ii) \text{ and } (P)$$

$$\Rightarrow (I) \text{ and } (II) \,.$$

For $p \neq 2$ the question is whether there are operators which satisfy both inequalities.

How about the p-Laplacian: Here we would have

$$A_\alpha^i = |\zeta|^{p-2}\zeta_\alpha^i$$

and so we need

$$|Du|^{(p-2)}|\sum_\alpha(\sum_i D^\alpha u^i v^i)\sum_i D^\alpha u^i v^i$$

$$\geq c(\sum_\alpha \sum_i D^\alpha u^i v^i)^p \,.$$

That is we can verify (S,ii), if $p > 2$, but not if $p < 2$.

On the other hand for $p < 2$ we have

$$|Du|^{(p-2)}|\sum_\alpha(\sum_i D^\alpha u^i v^i)$$

$$\leq c\sum_\alpha(\sum_i D^\alpha u^i v^i)^{\;p-1} .$$

That is we can verify (S,i) for $p < 2$, but not for $p > 2$.

Question: What are good additional conditions on u to provide regularity if $p \neq 2$, just boundedness seems not be enough.

Going back to the above estimates we did not use the term

$$\int \gamma(|Du|^p (M^2 - |v|^2)^\beta \eta^p) dx,$$

because of the power β instead of $\beta - 1$

If we do we get for the p-Laplacian

$$\int |Du|^{p-2} \sum_\alpha (\sum_i D_\alpha u^i v^i \sum_i v^i D_\alpha u^i)(M^2 - |v|^2)^{\beta-1} \eta^p dx$$

$$\leq C \int (M^2 - |v|^2)^{\beta+(p/2)} |D_\alpha \eta|^p dx$$

This allows us still to iterate for negative values of κ but we cannot quite reach zero as in the original proof where the iterations stalls out at zero. The gap is "jumped over" with the John-Nirenberg Lemma.

So one question I am working on right now is: are there any properties on u that allows us to determine how far we can jump?