# Mixed Problems for the Equation of Longitudinal Vibrations of a Rod Consisting of Two Segments with Different Densities and Elasticity Coefficients

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# Formulation of the problem

Consider the following mixed problems for the equation of longitudinal vibrations of rod

$$\rho(x)u_{tt}(x,t) = \frac{\partial}{\partial x}(k(x)u_x(x,t)) \qquad (1)$$

with piecewise constant Young's modulus $k(x)$ and density $\rho(x)$:

$$k(x) = \begin{cases} k_1 \text{ for } 0 \leqslant x < x_0, \\ k_2 \text{ for } x_0 \leqslant x \leqslant l; \end{cases} \qquad \rho(x) = \begin{cases} \rho_1 \text{ for } 0 \leqslant x < x_0, \\ \rho_2 \text{ for } x_0 \leqslant x \leqslant l; \end{cases}$$

where $x_0 \in (0,l), k_1, k_2, \rho_1, \rho_2$ are positive constants;
with zero initial conditions

$$u(x,0) = 0, u_t(x,0) = 0, \qquad (2)$$

and with one of the following sets of boundary conditions:

## Dirichlet and Neumann boundary conditions

$$\begin{cases} u(0,t) = \mu(t) \\ u(l,t) = \nu(t), \end{cases} \tag{3}$$

$\mu(t), \nu(t) \in W_2^1[0,T]$, $\mu(0) = 0$, $\nu(0) = 0$;

$$\begin{cases} u(0,t) = \mu(t) \\ u_x(l,t) = \nu(t), \end{cases} \tag{4}$$

$\mu(t) \in W_2^1[0,T]$, $\mu(0) = 0$, $\nu(t) \in L_2[0,T]$;

$$\begin{cases} u_x(0,t) = \mu(t) \\ u(l,t) = \nu(t), \end{cases} \tag{5}$$

$\mu(t) \in L_2[0,T]$, $\nu(t) \in W_2^1[0,T]$, $\nu(0) = 0$;

$$\begin{cases} u_x(0,t) = \mu(t) \\ u_x(l,t) = \nu(t), \end{cases} \tag{6}$$

$\mu(t), \nu(t) \in L_2[0,T]$.

# Review of existing results

Note that the explicit expressions for the solutions of the above-mentioned problems in case of identical travel times on both segments

$$\frac{l_1}{a_1} = \frac{l_2}{a_2}$$

were found in [Il'in, 2009], and in [Il'in, 2011] the optimization of Dirichlet boundary control was provided.
In case of equal impedances $\rho_1 k_1 = \rho_2 k_2$ the solutions of these mixed problems and optimal boundary control were obtained in [Il'in, Luferenko, 2009].

## Main definitions

In the rectangle $Q_T = [0 \leqslant x \leqslant l] \times [0 \leqslant t \leqslant T]$, we consider the class (introduced in [Il'in, 2000]) $\widehat{W}_2^1(Q_T)$ of functions $u(x,t)$ which are continuous in $Q_T$ and have generalized partial derivatives $u_x(x,t)$ and $u_t(x,t)$ belonging to the class $L_2(Q_T)$, to the class $L_2[0 \leqslant x \leqslant l]$ for all $t \in [0,T]$ and to the class $L_2[0 \leqslant t \leqslant T]$ for all $x \in [0,l]$.

**Definition 1.**

A generalized solution of the mixed problem for the wave equation (1) in the class $\widehat{W}_2^1(Q_T)$ with zero initial conditions (2) and with one of the sets of boundary conditions (3)-(6) is defined as a function $u(x,t) \in \widehat{W}_2^1(Q_T)$, which satisfies the condition $u(x,0) = 0$ for $0 \leqslant x \leqslant l$, satisfies the equality $u(0,t) = \mu(t)$ for $0 \leqslant t \leqslant T$ in case of boundary conditions $(3),(4)$, satisfies the equality $u(l,t) = \nu(t)$ for $0 \leqslant t \leqslant T$ in case of boundary conditions $(3),(5)$ and satisfies the integral identity

# The integral identity

$$\int_0^{x_0}\int_0^T \Big[k_1 u_x(x,t)\Phi_x(x,t) - \rho_1 u_t(x,t)\Phi_t(x,t)\Big]dxdt +$$

$$+\int_{x_0}^{l}\int_0^T \Big[k_2 u_x(x,t)\Phi_x(x,t) - \rho_2 u_t(x,t)\Phi_t(x,t)\Big]dxdt =$$

$$= \begin{cases} 0 \text{ in case of conditions } (3), \\ k_2 \int_0^T \nu(t)\Phi(l,t)dt \text{ in case of conditions } (4), \\ -k_1 \int_0^T \mu(t)\Phi(0,t)dt \text{ in case of conditions } (5), \\ k_2 \int_0^T \nu(t)\Phi(l,t)dt - k_1 \int_0^T \mu(t)\Phi(0,t)dt \text{ in case of conditions } (6), \end{cases}$$

# The description of test functions $\Phi(x,t)$

where $\Phi(x,t)$ is an arbitrary function of the class $W_2^1(Q_T)$, the traces of which satisfy the following conditions: $\Phi(x,T)=0$, $\Phi(0,t)=0$ in case of condition $u(0,t)=\mu(t)$, $\Phi(l,t)=0$ in case of condition $u(l,t)=\nu(t)$.

# Mixed problems with homogeneous boundary conditions at the right endpoint

Consider the following mixed problems for the wave equation (1) with zero initial conditions (2) and with homogeneous boundary conditions at the right endpoint ($\nu(t) \equiv 0$):

$$\begin{cases} u(0,t) = \mu(t) \\ u(l,t) = 0, \end{cases} \tag{7}$$

$$\begin{cases} u(0,t) = \mu(t) \\ u_x(l,t) = 0, \end{cases} \tag{8}$$

$$\begin{cases} u_x(0,t) = \mu(t) \\ u(l,t) = 0, \end{cases} \tag{9}$$

$$\begin{cases} u_x(0,t) = \mu(t) \\ u_x(l,t) = 0. \end{cases} \tag{10}$$

# Auxiliary definitions

For an arbitrary $T > 0$ we choose a positive integer $M_0$ such that $T \leqslant M_0 \min(\frac{x_0}{a_1}, \frac{l-x_0}{a_2})$ and introduce the following sets:
$\Omega_1 = \{(n,k) : (n,k) \in Z \times Z, n \geqslant 2, k \geqslant 0, n+k \leqslant M_0 - 1\}$ and
$\Omega_2 = \{(n,k) : (n,k) \in Z \times Z, n \geqslant 1, k \geqslant 1, n+k \leqslant M_0 - 1\}$.
Denote $a_1 = \sqrt{\frac{k_1}{\rho_1}}$, $a_2 = \sqrt{\frac{k_2}{\rho_2}}$. Then let us define the functions

$$\underline{\mu}(t) = \begin{cases} 0 \text{ for } t \leqslant 0, \\ \mu(t) \text{ for } t > 0; \end{cases} \qquad \underline{\nu}(t) = \begin{cases} 0 \text{ for } t \leqslant 0, \\ \nu(t) \text{ for } t > 0; \end{cases}$$

$$\widehat{\underline{\mu}}(t) = \int_0^t \underline{\mu}(\tau) d\tau, \ \widehat{\underline{\nu}}(t) = \int_0^t \underline{\nu}(\tau) d\tau.$$

Let us now present the explicit formulas for the generalized solutions of the above-mentioned problems (with homogeneous boundary conditions at the right endpoint).

## Proposition 1

For $T > 0$, $0 < x_0 < l$, $\mu(t) \in W_2^1[0,T]$, $\mu(0) = 0$ the mixed problem (1), (2), (7) has a generalized solution $u_1(x,t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$u_1(x,t) = \underline{\mu}(t - \frac{x}{a_1}) + \sum_{(n,k)\in\Omega_1} b_{n,k}[\underline{\mu}(t - \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l-x_0}{a_2}))-$$

$$-\underline{\mu}(t + \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l-x_0}{a_2}))] \text{ for } 0 \leqslant x \leqslant x_0,$$

$$u_1(x,t) = \sum_{(n,k)\in\Omega_2} c_{n,k}[\underline{\mu}(t - \frac{l-x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l-x_0}{a_2}))-$$

$$-\underline{\mu}(t + \frac{l-x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l-x_0}{a_2}))] \text{ for } x_0 \leqslant x \leqslant l.$$

# Proposition 2

For $T > 0$, $0 < x_0 < l$, $\mu(t) \in W_2^1[0, T]$, $\mu(0) = 0$ the mixed problem (1), (2), (8) has a generalized solution $u_2(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$u_2(x,t) = \underline{\mu}(t - \frac{x}{a_1}) + \sum_{(n,k)\in\Omega_1} b_{n,k}[\underline{\mu}(t - \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))-$$

$$-\underline{\mu}(t + \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))] \text{ for } 0 \leqslant x \leqslant x_0,$$

$$u_2(x,t) = \sum_{(n,k)\in\Omega_2} c_{n,k}[\underline{\mu}(t - \frac{l - x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))+$$

$$+\underline{\mu}(t + \frac{l - x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))] \text{ for } x_0 \leqslant x \leqslant l.$$

# Proposition 3

For $T>0$, $0<x_0<l$, $\mu(t)\in L_2[0,T]$ the mixed problem (1), (2), (9) has a generalized solution $u_3(x,t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$u_3(x,t)=-a_1\underline{\widehat{\mu}}(t-\frac{x}{a_1})+\sum_{(n,k)\in\Omega_1} b_{n,k}[\underline{\widehat{\mu}}(t-\frac{x}{a_1}-(n\frac{x_0}{a_1}+k\frac{l-x_0}{a_2}))+$$

$$+\underline{\widehat{\mu}}(t+\frac{x}{a_1}-(n\frac{x_0}{a_1}+k\frac{l-x_0}{a_2}))] \text{ for } 0\leqslant x\leqslant x_0,$$

$$u_3(x,t)=\sum_{(n,k)\in\Omega_2} c_{n,k}[\underline{\widehat{\mu}}(t-\frac{l-x}{a_2}-(n\frac{x_0}{a_1}+k\frac{l-x_0}{a_2}))-$$

$$-\underline{\widehat{\mu}}(t+\frac{l-x}{a_2}-(n\frac{x_0}{a_1}+k\frac{l-x_0}{a_2}))] \text{ for } x_0\leqslant x\leqslant l.$$

# Proposition 4

For $T > 0$, $0 < x_0 < l$, $\mu(t) \in L_2[0, T]$ the mixed problem (1), (2), (10) has a generalized solution $u_4(x,t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$u_4(x,t) = -a_1\underline{\widehat{\mu}}(t - \frac{x}{a_1}) + \sum_{(n,k)\in\Omega_1} b_{n,k}[\underline{\widehat{\mu}}(t - \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2})) +$$

$$+\underline{\widehat{\mu}}(t + \frac{x}{a_1} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))] \text{ for } 0 \leqslant x \leqslant x_0,$$

$$u_4(x,t) = \sum_{(n,k)\in\Omega_2} c_{n,k}[\underline{\widehat{\mu}}(t - \frac{l - x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2})) +$$

$$+\underline{\widehat{\mu}}(t + \frac{l - x}{a_2} - (n\frac{x_0}{a_1} + k\frac{l - x_0}{a_2}))] \text{ for } x_0 \leqslant x \leqslant l.$$

# Computing $b_{n,k}, c_{n,k}$: auxiliary definitions

Let us describe in more detail the algorithm of obtaining the coefficients $b_{n,k}, c_{n,k}$ in Proposition 4.

We introduce the following sets
$\Omega = \{(n,k) : (n,k) \in Z \times Z, n \geqslant 0, k \geqslant 0, n+k \leqslant M_0 - 2\}$,
$\overline{\Omega} = \{(n,k) : (n,k) \in Z \times Z, n \geqslant 0, k \geqslant 0, n+k \leqslant M_0 - 1\}$, and
note the embeddings $\Omega_1 \subset \overline{\Omega}$, $\Omega_2 \subset \overline{\Omega}$.
Define

$$\begin{cases} b_{n,k} = c_{n,k} = 0 \text{ for } (m,n) \notin \overline{\Omega}, \\ b_{0,k} = 0 \text{ for } k \geqslant 1, \\ c_{n,0} = 0 \text{ for } n \geqslant 0, \\ b_{0,0} = -a_1. \end{cases}$$

# Computing $b_{n,k}, c_{n,k}$: the algorithm

Then we find the values of $b_{n,k}, c_{n,k}$ for $(n,k) \in \overline{\Omega}$ using the following system of equations:

$$\begin{cases} b_{n-1,k} + b_{n+1,k} = c_{n,k-1} + c_{n,k+1} \\ a_1\rho_1(-b_{n-1,k} + b_{n+1,k}) = a_2\rho_2(c_{n,k-1} - c_{n,k+1}). \end{cases} \tag{11}$$

By solving for $n = 0, 1 \ldots M_0 - 2$ the system $(11)$ sequentially for $k = \overline{0, M_0 - n - 2}$, taking $b_{n+1,k}, c_{n,k+1}$for the unknowns, we express them through already known at each step coefficients $b_{n-1,k}, c_{n,k-1}$. (since the determinant of this system equals $a_1\rho_1 + a_2\rho_2 \neq 0$)

Thus we find all the $b_{n,k}, c_{n,k}$ for $(n,k) \in \overline{\Omega}$, and therefore (due to the embeddings) also all the $b_{n,k}$ for $(n,k) \in \Omega_1$ and all the $c_{n,k}$ for $(n,k) \in \Omega_2$.

Note that $u_i(x,t) = u_i(x, t, \mu(t), x_0, a_1, a_2), i = \overline{1,4}$.

# Mixed problems with homogeneous boundary conditions at the left endpoint

With the help of a linear change of variables, from the solutions obtained above, we will get the explicit formulas for the generalized solutions of the following mixed problems with homogeneous boundary conditions at the left endpoint ($\mu(t) \equiv 0$):

$$\begin{cases} u(0,t) = 0 \\ u(l,t) = \nu(t), \end{cases} \tag{12}$$

$$\begin{cases} u(0,t) = 0 \\ u_x(l,t) = \nu(t), \end{cases} \tag{13}$$

$$\begin{cases} u_x(0,t) = 0 \\ u(l,t) = \nu(t), \end{cases} \tag{14}$$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(l,t) = \nu(t). \end{cases} \tag{15}$$

**Proposition 5.** For $T > 0$, $0 < x_0 < l$, $\nu(t) \in W_2^1[0, T]$, $\nu(0) = 0$ the mixed problem (1), (2), (12) has a generalized solution $v_1(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$v_1(x, t) = u_1(l - x, t, \nu(t), l - x_0, a_2, a_1),$$

where $u_1(x, t, \mu(t), x_0, a_1, a_2)$ is the solution of the mixed problem (1), (2), (7).

**Proposition 6.** For $T > 0$, $0 < x_0 < l$, $\nu(t) \in L_2[0, T]$ the mixed problem (1), (2), (13) has a generalized solution $v_2(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$v_2(x, t) = u_3(l - x, t, -\nu(t), l - x_0, a_2, a_1),$$

where $u_3(x, t, \mu(t), x_0, a_1, a_2)$ is the solution of the mixed problem (1), (2), (9).

**Proposition 7.** For $T > 0$, $0 < x_0 < l$, $\nu(t) \in W_2^1[0, T]$, $\nu(0) = 0$ the mixed problem (1), (2), (14) has a generalized solution $v_3(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$v_3(x, t) = u_2(l - x, t, \nu(t), l - x_0, a_2, a_1),$$

where $u_2(x, t, \mu(t), x_0, a_1, a_2)$ is the solution of the mixed problem (1), (2), (8).

**Proposition 8.** For $T > 0$, $0 < x_0 < l$, $\nu(t) \in L_2[0, T]$ the mixed problem (1), (2), (15) has a generalized solution $v_4(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$v_4(x, t) = u_4(l - x, t, -\nu(t), l - x_0, a_2, a_1),$$

where $u_4(x, t, \mu(t), x_0, a_1, a_2)$ is the solution of the mixed problem (1), (2), (10).

# The main results

Now let us proceed to the statement of the main results. Note that all the explicit formulas for the generalized solutions of the original mixed problems (1), (2), (3)-(6) are obtained by summing the solutions of the mixed problems with homogeneous boundary conditions, described above.

**Theorem 1.** For $T > 0$, $0 < x_0 < l$, $\mu(t), \nu(t) \in W_2^1[0, T]$,$\mu(0) = 0$, $\nu(0) = 0$ the mixed problem (1), (2), (3) has a generalized solution $\widetilde{U_1}(x, t)$ in $\widehat{W_2^1}(Q_T)$, which is given by the following formula:

$$\widetilde{U_1}(x, t) = u_1(x, t) + v_1(x, t),$$

where $u_1(x, t), v_1(x, t)$are above-mentioned solutions of the mixed problems (1), (2), (7) and (1), (2), (12).

# The main results

**Theorem 2.** For $T > 0$, $0 < x_0 < l$, $\mu(t) \in W_2^1[0, T]$, $\mu(0) = 0$, $\nu(t) \in L_2[0, T]$ the mixed problem (1), (2), (4) has a generalized solution $\widetilde{U_2}(x, t)$ in $\widehat{W_2^1}(Q_T)$, which is given by the following formula:

$$\widetilde{U_2}(x, t) = u_2(x, t) + v_2(x, t),$$

where $u_2(x, t), v_2(x, t)$ are above-mentioned solutions of the mixed problems (1), (2), (8) and (1), (2), (13).

**Theorem 3.** For $T > 0$, $0 < x_0 < l$, $\mu(t) \in L_2[0, T]$, $\nu(t) \in W_2^1[0, T]$, $\nu(0) = 0$ the mixed problem (1), (2), (5)has a generalized solution $\widetilde{U_3}(x, t)$ in $\widehat{W_2^1}(Q_T)$, which is given by the following formula:

$$\widetilde{U_3}(x, t) = u_3(x, t) + v_3(x, t),$$

where $u_3(x, t), v_3(x, t)$ are above-mentioned solutions of the mixed problems (1), (2), (9) and (1), (2), (14).

# The main results

**Theorem 4.** For $T > 0$, $0 < x_0 < l$, $\mu(t), \nu(t) \in L_2[0, T]$ the mixed problem (1), (2), (6) has a generalized solution $\widetilde{U_4}(x, t)$ in $\widehat{W}_2^1(Q_T)$, which is given by the following formula:

$$\widetilde{U_4}(x, t) = u_4(x, t) + v_4(x, t),$$

where $u_4(x, t), v_4(x, t)$are above-mentioned solutions of the mixed problems (1), (2), (10) and (1), (2), (15).

**Theorem 5.** For $T > 0$, $0 < x_0 < l$ each of the mixed problems (1), (2), (3)-(6) can have at most one generalized solution in $\widehat{W}_2^1(Q_T)$.

The scheme of the proof of Theorem 5 can be found in [Ladyzhenskaya, 1953].

## Transverse vibrations of a string

Consider the equation of transverse vibrations of a string

$$\widetilde{\rho}(x)u_{tt}(x,t)=\widetilde{k}(x)u_{xx}(x,t), \qquad (16)$$

$$\widetilde{k}(x)=\begin{cases}\widetilde{k}_1 \text{ for } 0\leqslant x<x_0,\\ \widetilde{k}_2 \text{ for } x_0\leqslant x\leqslant l;\end{cases}\quad \widetilde{\rho}(x)=\begin{cases}\widetilde{\rho}_1 \text{ for } 0\leqslant x<x_0,\\ \widetilde{\rho}_2 \text{ for } x_0\leqslant x\leqslant l;\end{cases}$$

where $x_0\in(0,l),\widetilde{k}_1,\widetilde{k}_2,\widetilde{\rho}_1,\widetilde{\rho}_2$ are positive constants;

**Theorem 6.** For $T>0$, $0<x_0<l$ each of the mixed problems (16), (2), (3)-(6) has a unique generalized solution in $\widehat{W}_2^1(Q_T)$. The explicit formulas for these solutions are obtained by substituting

$$\rho(x)=\frac{\widetilde{\rho}(x)}{\widetilde{k}(x)},k(x)\equiv 1$$

into the expressions for the solutions of the mixed problems (1), (2), (3)-(6) respectively.

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**Thanks for your attention!**

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