Hardy inequality of higher order

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Abstract: We investigate the k-th order Hardy inequality (1.1) for functions satisfying rather general boundary conditions (1.2), show which of these conditions are admissible and derive sufficient, and necessary and sufficient conditions (for $0 < q < \infty$, $p > 1$) on u, v for (1.1) to hold.

1 Introduction

We will consider the k-th order Hardy inequality

$$\left(\int_a^b |f(x)|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_a^b |f^{(k)}(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{1.1}$$

with k a positive integer, where $-\infty < a < b < +\infty$, p and q are real parameters, $p > 1$, $q > 1$, and u, v are weight functions, i.e. functions measurable and positive a.e. in (a, b). We assume that the functions $f \in C^{k-1}[a,b]$, $f^{(k-1)} \in AC(a,b)$ satisfy the "boundary conditions"

$$\sum_{j=1}^{k}[\alpha_{i,j} f^{(j-1)}(a) + \beta_{i,j} f^{(j-1)}(b)] = 0 \qquad \text{for} \quad i = 1, ..., k, \tag{1.2}$$

with $\{\alpha_{i,j}\}_{i,j=1}^k$ and $\{\beta_{i,j}\}_{i,j=1}^k$ given real numbers.

The conditions (1.2) are reasonable since they allow to exclude, e.g., polynomials of order $\leqq k-1$, for which the right hand side in (1.1) vanishes while the left hand side can be positive. On the other hand, not every choice of $\alpha_{i,j}$, $\beta_{i,j}$ is admissible, which can be illustrated by the following simple example.

Example 1.1. We choose $k = 1$; then (1.2) has the form

$$\alpha f(a) + \beta f(b) = 0. \tag{1.3}$$

For $\alpha = -\beta \neq 0$, any non-zero *constant* function f satisfies (1.3), while the right hand side in (1.1) (with $k = 1$!) equals zero. Hence, the choice $\alpha + \beta = 0$ is not allowed.

Let us consider the boundary value problem (BVP) consisting of the ordinary differential equation

$$f^{(k)}(x) = g(x) \quad \text{on} \quad (a,\, b) \tag{1.4}$$

and of the boundary conditions (1.2).

If we denote by $G(x, y)$ the *Green function* of this BVP, then we have

$$f(x) = \int_a^b G(x,t) g(t)\, \mathrm{d}t \tag{1.5}$$

and we can rewrite (1.1) as the *weighted norm inequality*

$$\left(\int_a^b \left| \int_a^b G(x,t) g(t)\, \mathrm{d}t \right|^q u(x)\, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\int_a^b |g(x)|^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.6}$$

Consequently, we have to solve two problems:

Problem A. To find the Green function of the BVP (1.4) & (1.2), i.e. to determine the values $\alpha_{i,j}$, $\beta_{i,j}$ for which this BVP is uniquely solvable, and to determine the form of $G(x,t)$.

Problem B. With $G(x,t)$ given, to find conditions (sufficient or necessary and sufficient) on the weight functions u, v, for which (1.6) holds for every function g.

2 Problem A - the Green function

The general solution of equation (1.4) has the following form:

$$f(x) = \sum_{m=1}^{k} c_m x^{m-1} - \int_x^b \frac{(x-t)^{k-1}}{(k-1)!} g(t)\, \mathrm{d}t \tag{2.1}$$

with arbitrary coefficients $c_1, c_2, ..., c_k$. Then conditions (1.2) lead to the following system of linear equations for the unknown c_i's:

$$\sum_{m=1}^{k} c_m \left[\sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!} \left[\alpha_{i,j} a^{m-j} + \beta_{i,j} b^{m-j} \right] \right] \quad = \quad \sum_{j=1}^{k} \alpha_{i,j} \int_a^b \frac{(a-t)^{k-j}}{(k-j)!} g(t)\, \mathrm{d}t \tag{2.2}$$

for $i = 1, ..., k$.

The determinant of this system has the following form:

$$\Delta = \begin{vmatrix} \alpha_{1,1}+\beta_{1,1} & \dots & \sum\limits_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{1,j}a^{m-j}+\beta_{1,j}b^{m-j}\right] & \dots & \sum\limits_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{1,j}a^{k-j}+\beta_{1,j}b^{k-j}\right] \\ \downarrow i \vdots & \dots & \vdots & \dots. & \downarrow i \vdots \\ \alpha_{i,1}+\beta_{i,1} & \dots & \sum\limits_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{i,j}a^{m-j}+\beta_{i,j}b^{m-j}\right] & \dots & \sum\limits_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{i,j}a^{k-j}+\beta_{i,j}b^{k-j}\right] \\ \downarrow i \vdots & \dots & \vdots & \dots. & \downarrow i \vdots \\ \alpha_{k,1}+\beta_{k,1} & \dots & \sum\limits_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{k,j}a^{m-j}+\beta_{k,j}b^{m-j}\right] & \dots & \sum\limits_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{k,j}a^{k-j}+\beta_{k,j}b^{k-j}\right] \end{vmatrix}. \tag{2.3}$$

The system (2.2) has a unique solution if and only if its determinant Δ is not equal to zero, and hence, we have immediately

Theorem 2.1. *The k-th order Hardy inequality (1.1) is meaningful for functions f satisfying (1.2) if and only if $\Delta \neq 0$, where Δ is given by (2.3).*

Example 2.2. Let us consider the case mentioned in Example 1.1, i.e. $k = 1$ and the condition (1.3). Condition (1.3) is then condition (1.2) with $\alpha_{1,1} = \alpha$, $\beta_{1,1} = \beta$ and we have $\Delta = \alpha + \beta$. Hence the Hardy inequality (1.1) (for $k = 1$!) is meaningful for functions f satisfying (1.3) if and only if $\Delta \neq 0$.

Example 2.3. Some particular cases of the conditions (1.2) have been investigated earlier. In the book [2], the k-th order Hardy inequality (1.1) was considered under the boundary conditions

$$\begin{aligned} f^{(i)}(a) &= 0 \quad \text{for} \quad i \in M_0, \\ f^{(j)}(b) &= 0 \quad \text{for} \quad j \in M_1, \end{aligned} \tag{2.4}$$

where M_0, M_1 are subsets of the set $\mathbb{N}_k = \{0, 1, ..., k-1\}$. In [2], it was shown that the Hardy inequality (1.1) is meaningful if and only if the sets M_0, M_1 satisfy the so-called *Pólya condition*, i.e.

$$\sum_{i=0}^{r}(e_{0,i} + e_{1,i}) \geq r + 1, \qquad r = 0, 1, ..., k-1,$$

where

$$e_{\alpha,i} = \begin{cases} 1 & \text{if} \quad i \in M_\alpha \\ 0 & \text{if} \quad i \notin M_\alpha. \end{cases}$$

Hence, the condition $\Delta \neq 0$ can be called a Pólya condition for the general case (1.3).

Assuming that $\Delta \neq 0$ with Δ given by (2.3) and solving the system (2.2) we see that the components of its solution $[c_1, c_2, ..., c_k]$ are linear combinations of the integrals on the right hand side of (2.2). Hence we have the solution f of our BVP due to (2.1) in the form (1.5), i.e.

$$f(x) = \int_a^b G(x,t)g(t)\,\mathrm{d}t,$$

where the Green function is given by the formula

$$G(x,t) = \sum_{n=1}^{k} P_n(x) t^{n-1} - \frac{(x-t)^{k-1}}{(k-1)!} \chi_{(x,b)}(t), \tag{2.5}$$

where $P_n(x) = \sum_{m=1}^{k} a_{n,m} x^{m-1}$, $n = 1, ..., k$, are polynomials of order $\le k-1$. More precisely,

$$G(x,t) = \begin{cases} G_1(x,t) & \text{for} \quad a < t \le x < b, \\ G_2(x,t) & \text{for} \quad a < x < t < b, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} G_1(x,t) &= \sum_{n=1}^{k} P_n(x) t^{n-1} \qquad (2.7) \\ G_2(x,t) &= \sum_{n=1}^{k} P_n(x) t^{n-1} - \frac{(x-t)^{k-1}}{(k-1)!}, \end{aligned}$$

i.e.

$$G_2(x,t) = \sum_{n=1}^{k} Q_n(x) t^{n-1} \tag{2.8}$$

with

$$Q_n(x) = P_n(x) + \frac{(-1)^n}{(k-1)!} \binom{k-1}{n-1} x^{k-n}.$$

Consequently, the Green function is fully described and the problem A is solved.

3 Problem B – the Hardy inequality

In the sequel, we will suppose that $\Delta \ne 0$ for Δ from (2.3).

3.1 Sufficient conditions

Since due to (2.6) we have

$$\int_a^b G(x,t) g(t) \, \mathrm{d}t = \int_a^x G_1(x,t) g(t) \, \mathrm{d}t + \int_x^b G_2(x,t) g(t) \, \mathrm{d}t,$$

it is

$$\begin{aligned} &\int_a^b \left| \int_a^b G(x,t) g(t) \, \mathrm{d}t \right|^q u(x) \, \mathrm{d}x \\ \le \ & 2^{q-1} \left[\int_a^b \left| \int_a^x G_1(x,t) g(t) \, \mathrm{d}t \right|^q u(x) \, \mathrm{d}x + \int_a^b \left| \int_x^b G_2(x,t) g(t) \, \mathrm{d}t \right|^q u(x) \, \mathrm{d}x \right] \end{aligned}$$

and if we solve *two* Hardy-type inequalities

$$\left(\int_a^b \Big|\int_a^x G_1(x,t)g(t)\,\mathrm{d}t\Big|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_1\left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.1}$$

$$\left(\int_a^b \Big|\int_x^b G_2(x,t)g(t)\,\mathrm{d}t\Big|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_2\left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.2}$$

we have obviously solved the inequality (1.6).

Let us consider (3.1). Due to (2.7), inequality (3.1) will be satisfied if there will be

$$\int_a^b \Big|\int_a^x P_n(x)t^{n-1}g(t)\,\mathrm{d}t\Big|^q u(x)\,\mathrm{d}x \leq C_{1,n}\left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{q}{p}} \tag{3.3}$$

for $n = 1, 2, ..., k$. If we denote $h(t) = g(t)t^{n-1}$, we can rewrite (3.3) as

$$\int_a^b \Big|\int_a^x h(t)\,\mathrm{d}t\Big|^q |P_n(x)|^q\, u(x)\,\mathrm{d}x \leq C_{1,n}\left(\int_a^b |h(x)|^p x^{-(n-1)p} v(x)\,\mathrm{d}x\right)^{\frac{q}{p}}. \tag{3.4}$$

But this is the classical Hardy inequality for the function h with weight functions $U(x) = |P_n(x)|^q\, u(x)$, $V(x) = x^{-(n-1)p}v(x)$, and this inequality holds for $1 < p \leq q < \infty$ if and only if the function

$$A_{M,n}(x) = \left(\int_x^b u(t)\,|P_n(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)t^{(n-1)p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \tag{3.5}$$

is bounded, while for the case $1 < q < p < \infty$, the necessary and sufficient condition reads

$$B_{M,n} = \left(\int_a^b \left(\int_x^b u(t)\,|P_n(t)|^q\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_a^x v^{1-p'}(t)t^{(n-1)p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} v^{1-p'}(x)x^{(n-1)p'}\,\mathrm{d}x\right)^{\frac{1}{r}} < \infty \tag{3.6}$$

here $p' = \frac{p}{p-1}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. (For details, see, e.g. [2].)

Now, let us consider (3.2). Analogously as in the foregoing case, (3.2) will be satisfied, if - due to (2.8) - the following Hardy inequality for the function h with weight functions $U(x) = |Q_n(x)|^q u(x)$, $V(x) = x^{-(n-1)p}v(x)$ will be satisfied:

$$\int_a^b \Big|\int_x^b h(t)\,\mathrm{d}t\Big|^q |Q_n(x)|^q\, u(x)\,\mathrm{d}x \leq C_{2,n}\left(\int_a^b |h(x)|^p x^{-(n-1)p} v(x)\,\mathrm{d}x\right)^{\frac{q}{p}}. \tag{3.7}$$

In this case, the boundedness of the function

$$\tilde{A}_{M,n}(x) = \left(\int_a^x u(t)\,|Q_n(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^b v^{1-p'}(t)t^{(n-1)p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \tag{3.8}$$

for $1 < p \le q < \infty$ or the finiteness of the number

$$\tilde{B}_{M,n} = \left(\int_a^b \left(\int_a^x u(t)\, |Q_n(t)|^q \,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_x^b v^{1-p'}(t) t^{(n-1)p'} \,\mathrm{d}t \right)^{\frac{r}{q'}} v^{1-p'}(x) x^{(n-1)p'} \,\mathrm{d}x \right)^{\frac{1}{r}} \tag{3.9}$$

for $1 < q < p < \infty$ is necessary and sufficient for (3.7) to hold.

Consequently, we have found a number of *sufficient* conditions of the validity of the k-th order Hardy inequality (1.1):

Theorem 3.1. *Let be $1 < p, q < \infty$. For $k \in \mathbb{N}$, let $n = 1, 2, ..., k$. Let $P_n(x)$ and $Q_n(x)$ be the polynomials from (2.7) and (2.8), respectively. Let $A_{M,n}(x)$ and $\tilde{A}_{M,n}(x)$ be defined by (3.5) and (3.8), respectively, and $B_{M,n}$ and $\tilde{B}_{M,n}$ by (3.6) and (3.9), respectively. Then the k-th order Hardy inequality (1.1) holds for functions f satisfying the boundary conditions (1.2) if the weight functions u, v satisfy for $n = 1, 2, ..., k$ the conditions*

$$\sup_{x \in (a,\, b)} A_{M,n}(x) < \infty, \qquad \sup_{x \in (a,\, b)} \tilde{A}_{M,n}(x) < \infty \tag{3.10}$$

in the case $1 < p \le q < \infty$, or

$$B_{M,n} < \infty, \qquad \tilde{B}_{M,n} < \infty \tag{3.11}$$

in the case $1 < q < p < \infty$.

3.2 Necessary and sufficient conditions

The Hardy inequality of higher order is, as we have seen, closely connected with the Hardy-type inequality (1.6). This inequality with rather general kernels $K(x, t)$ was investigated by many authors, see e.g. [2]. Here, we use the fact that $K(x, t)$ is a Green function and we assume that $1 < p < \infty$, $q > 0$ and that

$$u,\, v^{1-p'} \in L^1_{loc}(a,\, b). \tag{3.12}$$

Let us denote Δ_1 and Δ_2 the closed triangles $\{(x, t) : a \le t \le x \le b\}$ and $\{(x, t) : a \le x \le t \le b\}$, respectively. Due to (2.6), (2.7) and (2.8), we have

$$G_i \in C(\Delta_i), \quad i = 1, 2. \tag{3.13}$$

Furthermore, suppose that

$$\left. \begin{array}{l} G_1(x, a), \quad G_1(b, t), \quad G_2(a, t), \quad G_2(x, b) \\ \text{do not vanish } \textit{identically} \text{ in (a,b).} \end{array} \right\} \tag{3.14}$$

Theorem 3.2. *Let $1 < p < \infty$, $q > 0$ and suppose (3.12), (3.13) and (3.14). Then the Hardy-type inequality (1.6) holds if and only if*

$$u, v^{1-p'} \in L^1(a, b). \tag{3.15}$$

Proof. **Necessity:** Suppose that (1.6) holds.
(i) Due to (3.14), there exists a point $t_a \in (a,b)$ such that $G_2(a,t_a) \neq 0$. Consequently, there exists $\varepsilon > 0$ such that $|G(x,t)| = |G_2(x,t)| \geq C_a > 0$ for all $(x,t) \in (a, a+\varepsilon) \times (t_a - \varepsilon, t_a + \varepsilon)$. Here we suppose that $[t_a - \varepsilon, t_a + \varepsilon] \subset (a,b)$. If we choose the test function as $f(t) = \chi_{(t_a-\varepsilon, t_a+\varepsilon)}(t) v^{1-p'}(t)$, we get from (1.6)

$$\begin{aligned}
C\left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p}} &= C\left(\int_a^b |f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\geq \left(\int_a^b \left|\int_a^b G(x,t) f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\int_a^b \left|\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t) G(x,t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\geq \left(\int_a^{a+\varepsilon} \left|\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t) G_2(x,t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\geq C_a \left(\int_a^{a+\varepsilon} \left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= C_a \left(\int_a^{a+\varepsilon} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t,
\end{aligned}$$

i.e.

$$\left(\int_a^{a+\varepsilon} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq \frac{C}{C_a}\left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty$$

due to (3.12). Together with (3.12), the last inequality implies that

$$\int_a^c u(x)\,\mathrm{d}x < \infty \qquad \text{for every} \quad c < b,$$

which we denote as

$$u \in L^1_{loc}([a,\, b)). \tag{3.16}$$

(ii) Due to (3.14), there exists a point $t_b \in (a,b)$ such that $G_1(b,t_b) \neq 0$ and $|G(x,t)| = |G_2(x,t)| \geq C_b > 0$ for all $(x,t) \in (b-\varepsilon, b) \times (t_b - \varepsilon, t_b + \varepsilon)$. The choice $f(t) = v^{1-p'}(t)\chi_{(t_b-\varepsilon, t_b+\varepsilon)}(t)$ leads analogously as in (i) to the estimate

$$\left(\int_{b-\varepsilon}^{b} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq \frac{C}{C_b}\left(\int_{t_b-\varepsilon}^{t_b+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty,$$

i.e.

$$\int_d^b u(x)\,\mathrm{d}x < \infty \qquad \text{for every} \quad d > a$$

or

$$u \in L^1_{loc}((a,b]).$$

This together with (3.16) and (3.12) gives $u \in L^1(a,b)$.

(iii) Due to (3.14), there exists a point $x_a \in (a,b)$ such that $G_1(x_a,a) \neq 0$ and $|G(x,t)| = |G_1(x,t)| \geq \hat{C}_a > 0$ for all $(x,t) \in (x_a-\varepsilon, x_a+\varepsilon) \times (a, a+\varepsilon)$. Let us choose a test function in (1.6) as

$$f(t) = \chi_{(a+\delta,a+\varepsilon)}(t) v^{1-p'}(t),$$

where $\delta \in (0,\varepsilon)$ is a parameter. Then we get that

$$\begin{aligned}
C\left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p}} &= C\left(\int_a^b |f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\geq \left(\int_a^b \left|\int_a^b G(x,t) f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\int_a^b \left|\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t) G(x,t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\geq \left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} \left|\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t) G_1(x,t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\geq \hat{C}_a\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} \left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \hat{C}_a\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t,
\end{aligned}$$

i.e.

$$\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} < \frac{C}{\hat{C}_a}.$$

This estimate holds for all $\delta \in (0,\varepsilon)$, and with δ tending to zero on the left hand side of the estimate we obtain

$$\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_{a}^{a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \leq \frac{C}{\hat{C}_a}$$

which implies that $v^{1-p'} \in L^1_{loc}([a,b))$.

(iv) Finally, we obtain analogously from $G_2(x_b,b) \neq 0$ that $v^{1-p'} \in L^1_{loc}((a,b])$, hence $v^{1-p'} \in L^1(a,b)$ and the necessity is proved.

Sufficiency: Using the boundedness of the function $G(x,t)$ (which follows from (3.13)), Hölder's inequality and (3.15), we estimate the left hand side of (1.6) as follows:

$$
\begin{aligned}
&\left(\int_a^b \left|\int_a^b G(x,t)g(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
\leq\ & \left(\int_a^b \left(\int_a^b |G(x,t)||g(t)|\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
\leq\ & C_1\left(\int_a^b \left(\int_a^b |g(t)|\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
=\ & C_1\left(\int_a^b u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \int_a^b |g(t)|\,\mathrm{d}t \\
=\ & C_1\left(\int_a^b u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_a^b |g(t)|v^{\frac{1}{p}}(t)v^{-\frac{1}{p}}(t)\,\mathrm{d}t\right) \\
\leq\ & C_1\left(\int_a^b u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_a^b v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \\
\leq\ & C\left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}
$$

□

Remark 3.3. We have considered the Hardy-type inequality (1.6) for the case that $G(x,t)$ was a Green function, i.e. $G_i(x,t)$ have been *polynomials.* It is obvious that we can repeat our approach for *any* function $G(x,t)$, which satisfies (3.13) and (3.14). Hence we have some new criteria of the validity of (1.6) for rather general kernels G.

Example 3.4. In Example 1.1, the first order Hardy inequality with boundary condition (1.3) was considered. It can be easily shown that the Green function has the form

$$
G(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta} & \text{for} \quad a < t < x < b, \\ -\frac{\beta}{\alpha+\beta} & \text{for} \quad a < x \leq t < b, \end{cases}
$$

where $\alpha+\beta \neq 0$. If $\alpha \neq 0$ and $\beta \neq 0$, then the conditions (3.14) are satisfied and we can use Theorem 3.2. According to this theorem, the Hardy inequality (1.6) holds if and only if

$$
u, v^{1-p'} \in L^1(a,b).
$$

Example 3.5. For simplicity let us assume for (a,b) the interval $(0,1)$ and consider the second order Hardy inequality

$$
\left(\int_0^1 |f(x)|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_0^1 |f''(x)|^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{3.17}
$$

Then boundary conditions (1.2) take the form

$$\begin{cases} \alpha_{1,1}f(0) + \alpha_{1,2}f'(0) + \beta_{1,1}f(1) + \beta_{1,2}f'(1) = 0 \\ \alpha_{2,1}f(0) + \alpha_{2,2}f'(0) + \beta_{2,1}f(1) + \beta_{2,2}f'(1) = 0. \end{cases} \tag{3.18}$$

This inequality was considered in [4] and the corresponding Green function has the form:

$$G(x,t) = \begin{cases} \frac{1}{\Delta}(a + bx + ct + dxt) & \text{for} \quad 0 < t < x < 1; \\ \frac{1}{\Delta}(a + (b-\Delta)x + (c+\Delta)t + dxt) & \text{for} \quad 0 < x \le t < 1, \end{cases}$$

where

$$\Delta = \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}, \quad a = \begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix}, \quad b = \begin{vmatrix} \lambda_1 & \alpha_{1,2} \\ \lambda_2 & \alpha_{2,2} \end{vmatrix}, \quad c = \begin{vmatrix} \mu_1 & \alpha_{1,1} \\ \mu_2 & \alpha_{2,1} \end{vmatrix}, \quad d = \begin{vmatrix} \lambda_1 & \beta_{1,1} \\ \lambda_2 & \beta_{2,1} \end{vmatrix}$$

with $\lambda_i := \alpha_{i,1} + \beta_{i,1}$, $\mu_i := \alpha_{i,1} + \beta_{i,1} + \beta_{i,2}$, $\nu_i := \beta_{i,1} + \beta_{i,2}$, $i = 1, 2$. Notice, the Δ is corresponding determinant from (2.3).

Let us use Theorem 3.2; for this aim we consider the polynomials:

$$G(x,0) = \frac{a+bx}{\Delta}, \quad G(1,t) = \frac{(a+b)+(c+d)t}{\Delta},$$

$$G(x,1) = \frac{(a+c+\Delta)+(b+d-\Delta)x}{\Delta}, \quad G(0,t) = \frac{a+(c+\Delta)t}{\Delta}.$$

These polynomials satisfy conditions (3.14) if and only if

$$|a|+|b| \ne 0, \quad |a+b|+|c+d| \ne 0, \quad |a+c+\Delta|+|b+d-\Delta| \ne 0, \quad |a|+|c+\Delta| \ne 0, \tag{3.19}$$

and these conditions imply that *the second order Hardy inequality holds if and only if* $u, v^{1-p'} \in L^1(0,1)$.

If the condition (3.14) is violated, then Theorem 3.2 cannot be used. Nevertheless, in some cases, it is possible to use the following generalization

Theorem 3.6. *Suppose* $1 < p < \infty$, $q > 0$ *and the functions* $G_i(x,t)\,(i = 1,2)$ *are not identically equal to zero.*

(i) If the Hardy-type inequality (1.6) holds, then there exist polynomials $P_i(x), Q_i(t)$ $(i = 1,2)$ *on* $(a,\, b)$ *such that*

$$|Q_1|^{p'} v^{1-p'}, |P_2|^q u \in L^1_{loc}([a,b)) \quad \textit{and} \quad |Q_2|^{p'} v^{1-p'}, |P_1|^q u \in L^1_{loc}((a,b]), \tag{3.20}$$

and that the corresponding Green function $G(x,t)$ *can be written as*

$$G_i(x,t) = P_i(x)Q_i(t)\hat{G}_i(x,t), \quad i = 1,2, \tag{3.21}$$

where the functions $\hat{G}_1(x,t)$, $\hat{G}_2(x,t)$ *satisfy (3.14).*

If, moreover, $\hat{G}_i(a,a) \neq 0$, $\hat{G}_i(b,b) \neq 0$, then

(i-1) for $p \leq q$

$$\sup_{x\in(a,b)} A_i(a,b;x) < \infty, \quad i = 1,2, \tag{3.22}$$

where

$$A_1(a,b;x) := \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{1}{q}}, \tag{3.23}$$

$$A_2(a,b;x) := \left(\int_x^b |Q_2|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_a^x |P_2|^q u\,\mathrm{d}t\right)^{\frac{1}{q}}; \tag{3.24}$$

(i-2) for $q < p$

$$B_i(a,b) < \infty, \quad i = 1,2, \tag{3.25}$$

where

$$B_1(a,b) := \left(\int_a^b \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{r}}, \tag{3.26}$$

$$B_2(a,b) := \left(\int_a^b \left(\int_x^b |Q_2|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_a^x |P_2|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_2(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{r}}. \tag{3.27}$$

(ii) If there exist polynomials $P_i(x)$, $Q_i(t)$ on $(a,\, b)$ $(i = 1,2)$ such that (3.21) holds and the conditions (3.22) (for $p \leq q$), (3.25) (for $q < p$) are satisfied, then the Hardy-type inequality (1.6) holds.

Proof. **(i)** Let the Hardy-type inequality (1.6) hold, then the following inequality

$$\left(\int_a^{a+\varepsilon} \left|\int_a^{a+\varepsilon} G(x,t) f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\int_a^{a+\varepsilon} |f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{3.28}$$

also holds for arbitrary function $f \in L^p(v)$, which follows from (1.6) considered for the function $g(t) = f(t)\chi_{(a,\,a+\varepsilon)}(t)$ and then from the monotonicity of the outer integral on the left hand side of (1.6).

(i.1) If $G_2(a,t)$ does not vanish identically on (a,b) then the proof of the existence of the polynomial $P_1(t)$ follows from point (i) of the proof of Theorem 3.2, i.e. in this case $u \in L^1_{loc}([a,b))$ and the polynomial can be chosen as $P_2(x) \equiv 1$.

(i.2) If $G_2(a,t)$ vanishes on $(a,\, b)$ then there exists a positive integer α_2 such that $G_2(x,t) = (x-a)^{\alpha_2}\hat{G}_2(x,t)$, where $\hat{G}_2(a,t)$ does not vanish on $(a,\, b)$. Choosing $\varepsilon > 0$ in inequality (3.28) sufficiently small and repeating the calculations in point (i) of the proof of Theorem 3.2 we obtain

$$\left(\int_a^{a+\varepsilon} (x-a)^{\alpha_2 q} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq \frac{C}{C_a} \left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\,\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty$$

which implies that $(x-a)^{\alpha_2 q}u \in L^1_{loc}[a,b)$ and the polynomial can be chosen as $P_2(x) \equiv (x-a)^{\alpha_2}$.

(i.3) Similarly, we can prove that there exist nonnegative integers $\alpha_1, \beta_1, \beta_2$ such that

$$(b-x)^{\beta_2 q}u \in L^1_{loc}((a,b]), \quad (b-t)^{\beta_1 p'}v^{1-p'}(t) \in L^1_{loc}((a,b]), \quad (t-a)^{\alpha_1 p'}v^{1-p'} \in L^1_{loc}([a,b))$$

and the polynomials can choose as

$$P_1(x) \equiv (b-x)^{\beta_1}, \quad Q_1(t) \equiv (t-a)^{\alpha_1}, \quad Q_2(t) \equiv (b-t)^{\beta_2}.$$

From these it can be easily shown that the weight functions with these polynomials satisfy (3.20) and (3.21).

(i.4) Now we show that the conditions (3.22) and (3.25) are satisfied. Using (3.21) we rewrite (3.28) in the form

$$\begin{aligned}&\left(\int_a^{a+\varepsilon}\left|\int_a^x P_1(x)Q_1(t)\hat{G}_1(x,t)f(t)\,\mathrm{d}t + \int_x^{a+\varepsilon} P_2(x)Q_2(t)\hat{G}_2(x,t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\ \le\ & C\left(\int_a^{a+\varepsilon}|f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}\end{aligned}$$

and taking into account that $\hat{G}_i(a,a) \neq 0$ $(i=1,2)$ we obtain the following equivalent inequality

$$\begin{aligned}&\left(\int_a^{a+\varepsilon}\left|P_1(x)\hat{G}_1(a,a)\int_a^x Q_1(t)f(t)\,\mathrm{d}t + P_2(x)\hat{G}_2(a,a)\int_x^{a+\varepsilon} Q_2(t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\ \le\ & C\left(\int_a^{a+\varepsilon}|f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}\end{aligned}$$

for all $f \in L^p(v)$ and for sufficiently small $\varepsilon > 0$. Using Theorem 2.3 in [2] we obtain the following equivalent conditions on the interval $(a, a+\varepsilon)$

(for $p \le q$)

$$\sup_{x\in(a,a+\varepsilon)} A_i(a,a+\varepsilon) < \infty, \quad i=1,2; \tag{3.29}$$

(for $q < p$)

$$B_i(a,a+\varepsilon) < \infty, \quad i=1,2. \tag{3.30}$$

Similarly, we obtain the following conditions on the interval $(b-\varepsilon, b)$:

(for $p \le q$)

$$\sup_{x\in(a,a+\varepsilon)} A_i(b-\varepsilon,b) < \infty, \quad i=1,2; \tag{3.31}$$

(for $q < p$)

$$B_i(b-\varepsilon, b) < \infty, \quad i = 1, 2. \tag{3.32}$$

All these conditions and together with (3.20) imply that conditions (3.22) and (3.25) are satisfied:

Condition (3.22). Using (3.20) it is easy to show that the condition is satisfied if and only if there exist the limits

$$\limsup_{x\to a+} A_i(a, b; x) \qquad \text{and} \qquad \limsup_{x\to b-} A_i(a, b; x) \quad \text{i=1,2.}$$

Otherwise, the existence of these limits is equivalent to the existence of

$$\limsup_{x\to a+} A_i(a, a+\varepsilon; x) \qquad \text{and} \qquad \limsup_{x\to b-} A_i(b-\varepsilon, b; x) \quad \text{i=1,2.}$$

For the proof of this assertion, we only show the following equality, since the others can be proved analogously:

$$\begin{aligned}
& \limsup_{x\to a+} A_1(a, b; x) \\
= \; & \limsup_{x\to a+} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{1}{q}} \\
= \; & \limsup_{x\to a+} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_x^{a+\varepsilon} |P_1|^q u\,\mathrm{d}t + \int_{a+\varepsilon}^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{1}{q}} \\
= \; & \limsup_{x\to a+} \left[\big[A_1(a, a+\varepsilon; x)\big]^q + \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{q}{p'}} \int_{a+\varepsilon}^b |P_1|^q u\,\mathrm{d}t\right]^{\frac{1}{q}} \\
= \; & \limsup_{x\to a+} A_1(a, a+\varepsilon; x).
\end{aligned}$$

The existence of the limits follows from (3.29) and (3.31).

Condition (3.25). Here it is enough to prove $B_1(a, b) < \infty$, since the case $B_2(a, b) < \infty$ can be proved analogously. First we rewrite $B_1(a, b)$ in the form

$$\begin{aligned}
B_1(a, b)^r \;=\; & \int_a^{a+\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
& + \int_{a+\varepsilon}^{b-\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
& + \int_{b-\varepsilon}^{b} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
:=\; & I_1 + I_2 + I_3.
\end{aligned}$$

The boundedness of I_2 follows from (3.20). Moreover, (3.20) together with (3.30) implies that

$$\begin{aligned}
I_1 &= \int_a^{a+\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\le 2^{\frac{r}{q}-1}\Bigg[\int_a^{a+\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^{a+\varepsilon} |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\quad + \int_a^{a+\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_{a+\varepsilon}^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\Bigg] \\
&= 2^{\frac{r}{p}}\left[\big[B_1(a,a+\varepsilon)\big]^r + \left(\int_{a+\varepsilon}^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} \int_a^{a+\varepsilon} \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\right] \\
&= 2^{\frac{r}{p}}\left[\big[B_1(a,a+\varepsilon)\big]^r + \frac{p'}{r}\left(\int_{a+\varepsilon}^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_a^{a+\varepsilon} |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{p'}}\right] < \infty
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{b-\varepsilon}^b \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\le \int_{b-\varepsilon}^b \left(\int_{b-\varepsilon}^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\quad + \left(\int_a^{b-\varepsilon} |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \int_{b-\varepsilon}^b \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\le C_\varepsilon \int_{b-\varepsilon}^b \left(\int_{b-\varepsilon}^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&= C_\varepsilon \big[B_1(a,a+\varepsilon)\big]^r < \infty.
\end{aligned}$$

To get the last estimate we used that

$$\begin{aligned}
&\left(\int_a^{b-\varepsilon} |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \int_{b-\varepsilon}^b \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
\le\ & C_\varepsilon \int_{b-\varepsilon}^b \left(\int_{b-\varepsilon}^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x,
\end{aligned}$$

which follows from (3.20).

(ii) Using the boundedness of the polynomials $\hat{G}_i$ (i=1,2) and Theorem 2.3 in [2] we immediately get the proof.

The proof is complete. □

Example 3.7. Let us go back to Example 3.5. If, e.g., $|a| + |b| = 0$, then one of the conditions (3.19) is violated. In this case we proceed according to Theorem 3.6 where $G_1(x,t) = \frac{1}{\Delta}t(c + dx) = t(c + dx)\hat{G}_1(x,t)$, $\hat{G}_1(x,t) \equiv \frac{1}{\Delta}$, and $\hat{G}_1(0,0) \neq 0$, $\hat{G}_1(1,1) \neq 0$; if, moreover, $c + \Delta = 0$, then $G_2(x,t) = \frac{x(dt-\Delta)}{\Delta}\hat{G}_2(x,t)$ where $\hat{G}_2(x,t) \equiv \frac{1}{\Delta}$, and the Hardy inequality (3.17) holds for functions satisfying (3.18) if and only if (3.22) (for $p \leq q$) or (3.25) (for $q < p$) hold with $P_1(x) = c + dx$, $Q_1(t) = t$; $P_2(x) = x$, $Q_2(t) = dt - \Delta$.

The other cases of violation of (3.19) can be considered analogously.

Acknowledgement. The second author was supported by the Research Plan MSM 4977751301 of the Ministry of Education, Youth and Sports of the Czech Republic.

References

[1] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality - About its History and Some Related Results*, Vydavatelský Servis, Plzeň, 2007. 162 pp. ISBN: 978-80-86843-15-5.

[2] A. Kufner, L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. xviii+357pp. ISBN: 981-238-195-3.

[3] A. Kufner, *Higher order Hardy inequalities*, Collect.Math. **44** (1993), no. 1-3, 147-154.

[4] A. Kufner, *A remark of kth order Hardy inequalities.* (Russian) Tr. Mat. Inst. Steklova **248** (2005), Issled. po Teor. Funlkts. i Differ. Uravn., 144-152; translation in Proc. Steklov Inst. Math. **2005,** no. 1(248), 138-146.

[5] V. S. Rychkov, *Splitting of Volterra integral operators with degenerate kernels*, (Russian) Tr. Math. Inst. Steklova **214**(1997), Issled. po Teor. Differ. Funkts. Mnogikh Perem. i. ee Prilozh. 17, 267-285; translation in Proc. Steklov Inst. Math. **1996**, no. 3(214), 260-278.