FSDONA 8

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HIGHER ORDER HARDY INEQUALITIES

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(A) $r \geq 1$, $w > 0$ **a.e. in** (a, b)

$$L^p(w) := \{f; \quad \|f\|_{r,w} < \infty\}$$

$$\|f\|_{r,w} := \left(\int_a^b |f(t)|^r w(t) dt \right)^{\frac{1}{r}}$$

HARDY'S INEQUALITY

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$$f(a) = 0 \text{ or } f(b) = 0 \text{ or } f(c) = 0 \quad \text{for } c \in (a,b)$$

$$\alpha f(a) + \beta f(b) = 0 \quad \Longrightarrow \quad \alpha, \beta \neq 0, \quad \alpha + \beta \neq 0$$

$k \in N$; $k-$th order Hardy inequality

$$(H_k) \quad \left(\int_a^b |f(x)|^q u(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\int_a^b |f^{(k)}(x)|^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}}$$

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avoid $f(x)$ =polynomial of order $k-1 \Longrightarrow$

$$\sum_{j=1}^{k} [\alpha_{i,j} f^{(j-1)}(a) + \beta_{i,j} f^{(j-1)}(b)] = 0 \qquad \text{for} \quad i = 1, \ldots, k,$$

a) Successively (H_1) for f, f', f'', ..., $f^{(k-1)}$

Example $k = 2$

$$(H_2) \qquad \|f\|_{q,u} \leq C\|f''\|_{p,v}$$

a) Successively (H_1) for f, f', f'', ..., $f^{(k-1)}$

Example $k = 2$

$$(H_2) \qquad \|f\|_{q,u} \le C\|f''\|_{p,v}$$

together with two conditions

$$\|f\|_{q,u} \le C_1\|f'\|_{r,w}, \quad \|f'\|_{r,w} \le C_2\|f''\|_{p,v}$$

$$f(c) = 0 \qquad f'(d) = 0$$

$r, w = ?$

b) special cases

(A)

$$f(a) = f'(a) = f''(a) = \dots = f^{(k-1)}(a) = 0$$

$$f(x) = \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} g(t)dt \Longrightarrow f^{(k)}(x) = g(x)$$

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(H_k) leads to

$$\left(\int_a^b \left| \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} g(t)dt \right|^q u(x)\, \mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\int_a^b |g(x)|^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}}$$

Hardy type inequality

$$(H_T) \quad \left(\int_a^b \left| \int_a^x K(x,t) g(t) dt \right|^q u(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}}$$

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Necessary and sufficient conditions:

Hardy type inequality

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Necessary and sufficient conditions:

V. STEPANOV for $K(x,t) = (x-t)^\alpha$

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$$(H_T) \quad \left(\int_a^b \left| \int_a^x K(x,t) g(t) dt \right|^q u(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\int_a^b |g(x)|^p v(x) \, \mathrm{d}x \right)^{\frac{1}{p}}$$

Necessary and sufficient conditions:

V. STEPANOV for $K(x,t) = (x-t)^\alpha$

R. OINAROV for general nonnegative kernels $K(x,t) \geq 0$

(B) H.P. HEINIG

$$f^{(i)}(a) = 0, \quad i = 0, 1, ..., m-1, \qquad f^{(j)}(a) = 0, \quad j = m, m+1, ..., k-1$$

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$$(H_T) \Longrightarrow$$

$$\int_a^b \left| \int_a^x A(x)B(t)g(t)dt \right|^q u(x)\,\mathrm{d}x = \int_a^b \left| \int_a^x B(t)g(t)dt \right|^q |A(x)|^q u(x)\,\mathrm{d}x$$

$$\leq C \left(\int_a^b |B(t)g(t)|^p B(t)^{-p} v(t)\,\mathrm{d}t \right)^{\frac{q}{p}} = \left(\int_a^b |g(t)|^p v(t)\,\mathrm{d}t \right)^{\frac{q}{p}}$$

New weights $U(x) = A^p(x)u(x), \; V(t) = B^{-p}(t)v(t)$

(C) G. SINNAMON

$$M_a, M_b \subset \{0, 1, ..., k-1\} := N_k$$

$$card M_a + card M_b = k$$

$$f \in \{f^{(i)}(a) = 0, i \in M_a; \quad f^{(j)}(b) = 0, j \in M_b\}$$

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PÓLYA CONDITION

E - incidence matrix

$$E = \begin{pmatrix} e_{a1}, & e_{a2}, & ..., & e_{ak} \\ e_{b1}, & e_{b2}, & ..., & e_{bk} \end{pmatrix}.$$

$e_{ai} = 1$ for $i \in M_a$ $e_{bj} = 1$ for $j \in M_b$, otherwise $e_{c\gamma} = 0$

$$\sum_{i=1}^{r} (e_{a,i} + e_{b,i}) \geq r, \qquad r = 1, ..., k$$

Examples:

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Here (H_K) leads to (H_T) with a **factorized kernel**

$$K(x,t) \approx A(x)B(t)$$

for $(a = 0, b = 1)$ $A(x) = x^\alpha(1-x)^\beta$, $B(t) = t^\gamma(1-t)^\delta$

The general case:

Solve the boundary value problem

$$(BVP) \qquad \begin{cases} f^{(k)}(x) = g(x) \quad \text{on} \quad (a,\ b) \\ \sum_{j=1}^{k} [\alpha_{i,j} f^{(j-1)}(a) + \beta_{i,j} f^{(j-1)}(b)] = 0 \qquad \text{for} \quad i = 1, ..., k, \end{cases}$$

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Denote $G(x,t)$ the *Green function*, then

$$f(x) = \int_a^b G(x,t) g(t) \, \mathrm{d}t$$

and (H_k) leads to (H_T) with kernel $K(x,t) = G(x,t)$

TWO PROBLEMS

A:

For what α_{ij}, β_{ij} is (BVP) uniquely solvable and how looks $G(x, t)$

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For what α_{ij}, β_{ij} is (BVP) uniquely solvable and how looks $G(x,t)$

B:

Under what conditions on u, v inequality (H_T) holds

Solution of the differential equation:

$$f(x) = \sum_{m=1}^{k} c_m x^{m-1} - \int_x^b \frac{(x-t)^{k-1}}{(k-1)!} g(t)\,\mathrm{d}t$$

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$$\sum_{m=1}^{k} c_m \left[\sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!} \left[\alpha_{i,j} a^{m-j} + \beta_{i,j} b^{m-j} \right] \right] = \sum_{j=1}^{k} \alpha_{i,j} \int_a^b \frac{(a-t)^{k-j}}{(k-j)!} g(t)\,\mathrm{d}t$$

for $i = 1, \dots, k$. This system is uniquely solvable, if determinant of the system Δ is different from zero

DETERMINANT $\Delta =$

$$\left| \begin{array}{ccccc} \alpha_{1,1}+\beta_{1,1} & .. & \sum\limits_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\left[\alpha_{1,j}a^{m-j}+\beta_{1,j}b^{m-j}\right] & .. & \sum\limits_{j=1}^{k}\frac{(k-1)!}{(k-j)!}\left[\alpha_{1,j}a^{k-j}+\beta_{1,j}b^{k-j}\right] \\ \downarrow i\vdots & .. & \vdots & .. & \downarrow i\vdots \\ \alpha_{i,1}+\beta_{i,1} & .. & \sum\limits_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\left[\alpha_{i,j}a^{m-j}+\beta_{i,j}b^{m-j}\right] & .. & \sum\limits_{j=1}^{k}\frac{(k-1)!}{(k-j)!}\left[\alpha_{i,j}a^{k-j}+\beta_{i,j}b^{k-j}\right] \\ \downarrow i\vdots & .. & \vdots & .. & \downarrow i\vdots \\ \alpha_{k,1}+\beta_{k,1} & .. & \sum\limits_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\left[\alpha_{k,j}a^{m-j}+\beta_{k,j}b^{m-j}\right] & .. & \sum\limits_{j=1}^{k}\frac{(k-1)!}{(k-j)!}\left[\alpha_{k,j}a^{k-j}+\beta_{k,j}b^{k-j}\right] \end{array} \right|$$

Determinant $\Delta \neq 0$

Example: $k = 1$

$$\Delta = \alpha + \beta$$

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Example: $k = 2$

- $f(a) = f(b) = 0 \quad \Delta = \alpha_{11}\beta_{21} = 1 \qquad \begin{pmatrix} 1, & 0 \\ 1, & 0 \end{pmatrix}$

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- $f(a) = f'(b) = 0 \quad \Delta = 1 \qquad \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$

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$$\Delta = \alpha + \beta$$

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Green function

$$G(x,t) = \sum_{n=1}^{k} P_n(x) t^{n-1} - \frac{(x-t)^{k-1}}{(k-1)!} \chi_{(x,\,b)}(t)$$

$$P_n(x) = \sum_{m=1}^{k} a_{n,m} x^{m-1}, \quad n = 1, \ldots, k$$

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$$G(x,t) = \begin{cases} G_1(x,t) = \sum_{n=1}^{k} P_n(x) t^{n-1} & \text{for} \quad a < t \le x < b, \\ G_2(x,t) = \sum_{n=1}^{k} Q_n(x) t^{n-1} & \text{for} \quad a < x < t < b, \end{cases}$$

$$Q_n(x) = P_n(x) + \frac{(-1)^n}{(k-1)!} \binom{k-1}{n-1} x^{k-n}.$$

(H_T)

$$\int_a^b \left| \sum_{n=1}^{k} \int_a^x P_n(x) t^{n-1} g(t)\,\mathrm{d}t + \sum_{n=1}^{k} \int_x^b Q_n(x) t^{n-1} g(t)\,\mathrm{d}t \right|^q u(x)\,\mathrm{d}x$$
$$\leq C^q \left(\int_a^b |g(t)|^p v(t)\,\mathrm{d}t \right)^{\frac{q}{p}}.$$

Sufficient conditions

$$\int_a^b \left| \int_a^x P_n(x) t^{n-1} g(t)\,\mathrm{d}t \right|^q u(x)\,\mathrm{d}x \le C_{1,n} \left(\int_a^b |g(x)|^p v(x)\,\mathrm{d}x \right)^{\frac{q}{p}}$$

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$$\int_a^b \left| \int_a^x h(t)\, \mathrm{d}t \right|^q |P_n(x)|^q\, u(x)\, \mathrm{d}x \le C_{1,n} \left(\int_a^b |h(x)|^p x^{-(n-1)p} v(x)\, \mathrm{d}x \right)^{\frac{q}{p}}.$$

classical Hardy inequality for $h(t) = g(t) t^{n-1}$ with new weights

$$U(x) = |P_n(x)|^q\, u(x), \quad V(x) = x^{-(n-1)p} v(x)$$

Sufficient conditions

Conditions:

$$A_{M,n}(x) = \left(\int_x^b u(t) \, |P_n(t)|^q \, \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t) t^{(n-1)p'} \, \mathrm{d}t \right)^{\frac{1}{p'}}$$

for $1 < p \leq q < \infty$,

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for $1 < p \leq q < \infty$,

$$B_{M,n} = \left(\int_a^b \left(\int_x^b u(t)\, |P_n(t)|^q \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t) t^{(n-1)p'} \, \mathrm{d}t \right)^{\frac{r}{q'}} v^{1-p'}(x) x^{(n-1)p'} \, \mathrm{d}x \right)^{\frac{1}{r}};$$

for $0 < q < p < \infty$, $p > 1$

Analogously

$$\int_a^b \left| \int_x^b h(t)\,\mathrm{d}t \right|^q |Q_n(x)|^q \, u(x)\,\mathrm{d}x \le C_{2,n} \left(\int_a^b |h(x)|^p x^{-(n-1)p} v(x)\,\mathrm{d}x \right)^{\frac{q}{p}}$$

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Conditions:

$$\tilde{A}_{M,n}(x) = \left(\int_a^x u(t)\,|Q_n(t)|^q\ \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_x^b v^{1-p'}(t) t^{(n-1)p'}\ \mathrm{d}t \right)^{\frac{1}{p'}}$$

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for $0 < q < p < \infty$, $p > 1$

Theorem 1:

Let be $1 < p, q < \infty$. For $k \in \mathbb{N}$, let $n = 1, 2, ..., k$. Then the k-th order Hardy inequality (H_T) holds for functions f satisfying the boundary conditions if the weight functions u, v satisfy for $n = 1, 2, ..., k$ the conditions

$$\sup_{x \in (a,\, b)} A_{M,n}(x) < \infty, \qquad \sup_{x \in (a,\, b)} \tilde{A}_{M,n}(x) < \infty$$

in the case $1 < p \leq q < \infty$, or

$$B_{M,n} < \infty, \qquad \tilde{B}_{M,n} < \infty$$

in the case $1 < q < p < \infty$.

NECESSARY CONDITIONS

Under additional conditions on u, v

$$A_{M,n}(x) \quad \text{bounded}, \qquad \tilde{A}_{M,n}(x) \quad \text{bounded}$$

are necessary for (H_T).

class $B_\delta = \{w : \quad \int_I w(x)dx \leq C(w) \int_{I/2} w(x)dx\}$

$$\begin{array}{ll} 1 < p < 2 & v^{1-p'} \in B_\delta \\ 2 < p < \infty & u^{1-q'} \in B_\delta . \end{array}$$

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$$1 < p < 2 \qquad v^{1-p'} \in B_\delta$$
$$2 < p < \infty \qquad u^{1-q'} \in B_\delta.$$

We use a result of Rychkov ($p = q = 2$) extended to $1 < p \leq q < \infty$

$u,\ v^{1-p'} \in L^1_{loc}(a,\ b)$

Theorem 2:

If

$$K(a,t),\ K(b,t),\ K(x,a),\ K(x,b) \quad \text{nonzero functions on} \quad (a,b)$$

then $u,\ v^{1-p'} \in L^1(a,\ b)$ are necessary and sufficient for $1 < p \leq q < \infty$.

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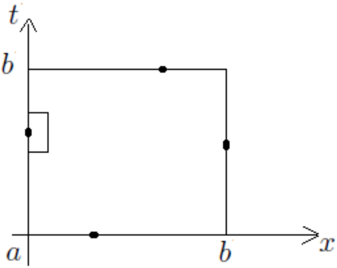

Examples:

Example $k = 1$. $\alpha f(a) + \beta f(b) = 0, \quad \alpha + \beta \neq 0, \quad \alpha, \beta \neq 0$

$$K(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta} & a < t < x < b \\ -\frac{\beta}{\alpha+\beta} & a < x < t < b \end{cases}$$

Examples:

Example $k = 2$. $\Delta \neq 0, \quad (a, b) = (0, 1)$.

$$\begin{cases} \alpha_{1,1} f(0) + \alpha_{1,2} f'(0) + \beta_{1,1} f(1) + \beta_{1,2} f'(1) = 0 \\ \alpha_{2,1} f(0) + \alpha_{2,2} f'(0) + \beta_{2,1} f(1) + \beta_{2,2} f'(1) = 0, \end{cases}$$

Examples:

Example $k = 2$. $\quad \Delta \neq 0, \quad (a, b) = (0, 1)$.

$$\begin{cases} \alpha_{1,1} f(0) + \alpha_{1,2} f'(0) + \beta_{1,1} f(1) + \beta_{1,2} f'(1) = 0 \\ \alpha_{2,1} f(0) + \alpha_{2,2} f'(0) + \beta_{2,1} f(1) + \beta_{2,2} f'(1) = 0, \end{cases}$$

$$K(x,t) = \begin{cases} \frac{1}{\Delta}(a + bx + ct + dxt) & \text{for} \quad 0 < t < x < 1; \\ \frac{1}{\Delta}(a + (b-\Delta)x + (c+\Delta)t + dxt) & \text{for} \quad 0 < x \le t < 1, \end{cases}$$

where $\Delta = \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}$,

$$a = \begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix}, \quad b = \begin{vmatrix} \lambda_1 & \alpha_{1,2} \\ \lambda_2 & \alpha_{2,2} \end{vmatrix}, \quad c = \begin{vmatrix} \mu_1 & \alpha_{1,1} \\ \mu_2 & \alpha_{2,1} \end{vmatrix}, \quad d = \begin{vmatrix} \lambda_1 & \beta_{1,1} \\ \lambda_2 & \beta_{2,1} \end{vmatrix}$$

with $\lambda_i := \alpha_{i,1} + \beta_{i,1}, \quad \mu_i := \alpha_{i,1} + \beta_{i,1} + \beta_{i,2}, \quad \nu_i := \beta_{i,1} + \beta_{i,2}, \quad i = 1, 2.$

Theorem 3: $1 < p < \infty$, $q > 0$ and $G_i(x,t)\,(i = 1,2)$ are not identically zero functions.

(i) If the Hardy-type inequality holds, then there exist polynomials $P_i(x), Q_i(t)$ $(i = 1,2)$ on $(a,\ b)$ such that

$$|Q_1|^{p'} v^{1-p'}, |P_2|^q u \in L^1_{loc}([a,b)) \quad \text{and} \quad |Q_2|^{p'} v^{1-p'}, |P_1|^q u \in L^1_{loc}((a,b]), \qquad (1)$$

and

$$G_i(x,t) = P_i(x) Q_i(t) \hat{G}_i(x,t), \quad i = 1,2. \qquad (2)$$

If, moreover, $\hat{G}_i(a,a) \neq 0$, $\hat{G}_i(b,b) \neq 0$, then

(i-1) for $p \leq q$

$$\sup_{x \in (a,b)} A_i(a,b;x) < \infty, \quad i = 1,2, \qquad (3)$$

(i-2) for $q < p$

$$B_i(a,b) < \infty, \quad i = 1,2, \qquad (4)$$

(ii) If there exist polynomials $P_i(x)$, $Q_i(t)$ on $(a,\ b)$ $(i = 1,2)$ such that (2) holds and the conditions (3) (for $p \leq q$), (4) (for $q < p$) are satisfied, then the Hardy-type inequality holds.

where

for $1 < p \le q < \infty$

$$A_1(a,b;x) := \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{1}{q}},$$

$$A_2(a,b;x) := \left(\int_x^b |Q_2|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_a^x |P_2|^q u\,\mathrm{d}t\right)^{\frac{1}{q}};$$

and for $1 < p \le q < \infty$

$$B_1(a,b) := \left(\int_a^b \left(\int_a^x |Q_1|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_1(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{r}},$$

$$B_2(a,b) := \left(\int_a^b \left(\int_x^b |Q_2|^{p'} v^{1-p'}\,\mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_a^x |P_2|^q u\,\mathrm{d}t\right)^{\frac{r}{q}} |Q_2(x)|^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{r}}.$$

Example:

If, e.g., $|a| + |b| = 0$, then $G_1(x,t) = \frac{1}{\Delta} t(c + dx) = t(c + dx)\hat{G}_1(x,t)$, $\hat{G}_1(x,t) \equiv \frac{1}{\Delta}$, and $\hat{G}_1(0,0) \neq 0$, $\hat{G}_1(1,1) \neq 0$; if, moreover, $c + \Delta = 0$, then $G_2(x,t) = \frac{x(dt-\Delta)}{\Delta} \hat{G}_2(x,t)$ where $\hat{G}_2(x,t) \equiv \frac{1}{\Delta}$, and the Hardy-type inequality holds if and only if (3) (for $p \leq q$) or (4) (for $q < p$) hold with $P_1(x) = c + dx$, $Q_1(t) = t$; $P_2(x) = x$, $Q_2(t) = dt - \Delta$.

The other cases of violation can be considered analogously.

THANK YOU!