Eigenvalues of Hille-Tamarkin operators and geometry of Banach function spaces

Thomas Kühn

Universität Leipzig, Germany

FSDONA-2011, Tabarz (Germany)
18–24 September 2011

The talk is based on a two joint papers with M. Mastyło (Poznań)

1. Weyl numbers and eigenvalues of abstract summing operators, J. Math. Anal. Appl. 369 (2010) 408–422.
2. Eigenvalues of Hille-Tamarkin operators and geometry of Banach function spaces, submitted (2011)

The talk is based on a two joint papers with M. Mastyło (Poznań)

1. Weyl numbers and eigenvalues of abstract summing operators, J. Math. Anal. Appl. 369 (2010) 408–422.
2. Eigenvalues of Hille-Tamarkin operators and geometry of Banach function spaces, submitted (2011)

Plan of the talk

1. Riesz operators and eigenvalues
2. Hille-Tamarkin operators
3. Geometry of Banach function spaces
4. Eigenvalue estimates
5. Optimality

1. Riesz operators and eigenvalues

1. Riesz operators and eigenvalues

- Definition. A bounded linear operator $T : X \to X$ in a complex Banach space X is called a Riesz operator, if for all complex numbers $\lambda \neq 0$ the operator $T - \lambda Id$ is Fredholm, i.e.
  (a) its kernel is finite-dimensional and
  (b) its range is closed and has finite codimension.

1. Riesz operators and eigenvalues

- Definition. A bounded linear operator $T : X \to X$ in a complex Banach space X is called a Riesz operator, if for all complex numbers $\lambda \neq 0$ the operator $T - \lambda Id$ is Fredholm, i.e.
  (a) its kernel is finite-dimensional and
  (b) its range is closed and has finite codimension.

- Examples.
  - compact operators
  - power-compact operators (i.e. T^n is compact for some $n \in \mathbb{N}$)

1. Riesz operators and eigenvalues

- Definition. A bounded linear operator $T : X \to X$ in a complex Banach space X is called a Riesz operator, if for all complex numbers $\lambda \neq 0$ the operator $T - \lambda Id$ is Fredholm, i.e.
  (a) its kernel is finite-dimensional and
  (b) its range is closed and has finite codimension.

- Examples.
  - compact operators
  - power-compact operators (i.e. T^n is compact for some $n \in \mathbb{N}$)

- Theorem. The spectrum $\sigma(T)$ of a Riesz operator T has no accumulation points (except possibly 0), and all non-zero $\lambda \in \sigma(T)$ are eigenvalues of finite algebraic multiplicity, i.e.

$$\mathrm{mult}(\lambda) := \dim\left(\bigcup_{n=1}^{\infty} \ker\Big((T - \lambda Id)^n\Big)\right) < \infty \,.$$

- Ordering of the eigenvalues (counting multiplicities)

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)| \geq \ldots \geq 0$$

If T has $< n$ eigenvalues, we set $\lambda_n(T) = \lambda_{n+1}(T) = \ldots = 0$.

- Ordering of the eigenvalues (counting multiplicities)

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)| \geq \ldots \geq 0$$

  If T has $< n$ eigenvalues, we set $\lambda_n(T) = \lambda_{n+1}(T) = \ldots = 0$.

- The general problem
  Find the optimal asymptotic eigenvalue behaviour of operators belonging to certain classes of operators, i.e.

$$|\lambda_n(T)| \preceq \ ??? \quad \text{or} \quad |\lambda_n(T)| \sim \ ??? \qquad \text{as} \quad n \to \infty$$

- Ordering of the eigenvalues (counting multiplicities)

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)| \geq \ldots \geq 0$$

  If T has $< n$ eigenvalues, we set $\lambda_n(T) = \lambda_{n+1}(T) = \ldots = 0$.

- The general problem
  Find the optimal asymptotic eigenvalue behaviour of operators belonging to certain classes of operators, i.e.

$$|\lambda_n(T)| \preceq \ ??? \quad \text{or} \quad |\lambda_n(T)| \sim \ ??? \qquad \text{as} \quad n \to \infty$$

- Our goal
  Solve this problem for a large class of integral operators, including the classical Hille-Tamarkin and weakly singular kernels.

  Tools
  – absolutely summing norms of operators
  – geometry of Banach spaces (cotype p and p-concavity)

- Let $1 \leq p < \infty$. The absolutely p-summing norm of an operator $T : X \to Y$ with respect to n vectors is defined as $\pi_p^{(n)}(T) = \inf c$ such that for all $x_1, ..., x_n \in X$

$$\underbrace{\left(\sum_{j=1}^{n} \|Tx_j\|^p\right)^{1/p}}_{\text{strong } \ell_p\text{-norm of } (Tx_j)} \leq c \underbrace{\sup_{\|a\|=1} \left(\sum_{j=1}^{n} |\langle x_j, a\rangle|^p\right)^{1/p}}_{:= w_p(x_j) \,=\, \text{weak } \ell_p\text{-norm of } (x_j)} .$$

$T \in \Pi_p(X, Y)$, if $\pi_p(T) := \sup_n \pi_p^{(n)}(T) < \infty$.

- Let $1 \leq p < \infty$. The absolutely p-summing norm of an operator $T : X \to Y$ with respect to n vectors is defined as $\pi_p^{(n)}(T) = \inf c$ such that for all $x_1, ..., x_n \in X$

$$\underbrace{\left(\sum_{j=1}^{n} \|Tx_j\|^p \right)^{1/p}}_{\text{strong } \ell_p\text{-norm of } (Tx_j)} \leq c \underbrace{\sup_{\|a\|=1} \left(\sum_{j=1}^{n} |\langle x_j, a \rangle|^p \right)^{1/p}}_{:= w_p(x_j) = \text{ weak } \ell_p\text{-norm of } (x_j)} .$$

$T \in \Pi_p(X, Y)$, if $\pi_p(T) := \sup_n \pi_p^{(n)}(T) < \infty$.

- absolutely $(p, 2)$-summing operators $(2 < p < \infty)$ are defined similarly, replacing $w_p(x_j)$ by $w_2(x_j)$

- Let $1 \leq p < \infty$. The absolutely p-summing norm of an operator $T : X \to Y$ with respect to n vectors is defined as $\pi_p^{(n)}(T) = \inf c$ such that for all $x_1, ..., x_n \in X$

$$\underbrace{\left(\sum_{j=1}^{n} \|Tx_j\|^p \right)^{1/p}}_{\text{strong } \ell_p\text{-norm of } (Tx_j)} \leq c \underbrace{\sup_{\|a\|=1} \left(\sum_{j=1}^{n} |\langle x_j, a\rangle|^p \right)^{1/p}}_{:= w_p(x_j) = \text{ weak } \ell_p\text{-norm of } (x_j)} .$$

  $T \in \Pi_p(X, Y)$, if $\pi_p(T) := \sup_n \pi_p^{(n)}(T) < \infty$.

- absolutely $(p, 2)$-summing operators $(2 < p < \infty)$ are defined similarly, replacing $w_p(x_j)$ by $w_2(x_j)$

- Classical results (due to König et al.)
  (i) $T \in \Pi_p(X, X) \Longrightarrow T$ is Riesz with eigenvalues in $\ell_{\max(p,2)}$
  (ii) $T \in \Pi_{p,2}(X, X) \Longrightarrow T$ is Riesz with eigenvalues in $\ell_{p,\infty}$

2. Hille-Tamarkin operators

2. Hille-Tamarkin operators

- $(\Omega, \mathcal{F}, \mu)$ (σ-finite) measure space

  $k : \Omega \times \Omega \to \mathbb{C}$ $(\mu \otimes \mu)$-measurable kernel

  $\curvearrowright$ integral operator $T_k f(x) = \int_\Omega k(x,t) f(t)\, d\mu(t)$

2. Hille-Tamarkin operators

- $(\Omega, \mathcal{F}, \mu)$ (σ-finite) measure space

  $k : \Omega \times \Omega \to \mathbb{C}$ $(\mu \otimes \mu)$-measurable kernel

  $\curvearrowright$ integral operator $T_k f(x) = \int_\Omega k(x,t) f(t)\, d\mu(t)$

- Classical Hille-Tamarkin kernels. $(1 \leq p < \infty \, , \, 1/p + 1/p' = 1)$

$$k \in L_p[L_{p'}], \text{i.e.} \quad \left(\int_\Omega \left(\int_\Omega |k(x,t)|^{p'} \, d\mu(t) \right)^{p/p'} d\mu(x) \right)^{1/p} < \infty$$

  T_k maps L_p into itself, even more: T_k is absolutely p-summing.
  $\curvearrowright$ $T_k : L_p \to L_p$ is Riesz with eigenvalues in $\ell_{\max(p,2)}$

2. Hille-Tamarkin operators

- $(\Omega, \mathcal{F}, \mu)$ (σ-finite) measure space

  $k : \Omega \times \Omega \to \mathbb{C}$ $(\mu \otimes \mu)$-measurable kernel

  $\curvearrowright$ integral operator $T_k f(x) = \int_\Omega k(x,t) f(t)\, d\mu(t)$

- Classical Hille-Tamarkin kernels. $(1 \leq p < \infty\ ,\ 1/p + 1/p' = 1)$

  $$k \in L_p[L_{p'}], \text{ i.e. } \quad \left(\int_\Omega \left(\int_\Omega |k(x,t)|^{p'}\, d\mu(t) \right)^{p/p'} d\mu(x) \right)^{1/p} < \infty$$

  T_k maps L_p into itself, even more: T_k is absolutely p-summing.
  $\curvearrowright$ $T_k : L_p \to L_p$ is Riesz with eigenvalues in $\ell_{\max(p,2)}$

- Remarks.
  - T is Hilbert-Schmidt in L_2 $\iff$ $T = T_k$ for some $k \in L_2[L_2]$
  - The kernel classes $L_p[L_{p'}]$ for different p's are incomparable!

- Let X be a Banach function space over $(\Omega, \mathcal{F}, \mu)$
  i.e. a Banach space $X \subset L_0(\Omega, \mathcal{F}, \mu)$ with the property:

  $f \in L_0$ and $g \in X$ s.t. $|f| \leq g \Longrightarrow f \in X$ and $\|f\|_X \leq \|g\|_X$

- Let X be a Banach function space over $(\Omega, \mathcal{F}, \mu)$
  i.e. a Banach space $X \subset L_0(\Omega, \mathcal{F}, \mu)$ with the property:

  $f \in L_0$ and $g \in X$ s.t. $|f| \leq g \Longrightarrow f \in X$ and $\|f\|_X \leq \|g\|_X$

- Köthe dual $X' := \{f \in L_0 : \int_\Omega f g \, d\mu$ exists $\forall g \in X\}$.
  - X' is always a closed subspace of the topological dual X^*
  - often $X' = X^*$, e.g. $L_p' = L_p^* = L_{p'}$ for $1 \leq p < \infty$

- Let X be a Banach function space over $(\Omega, \mathcal{F}, \mu)$
  i.e. a Banach space $X \subset L_0(\Omega, \mathcal{F}, \mu)$ with the property:

  $f \in L_0$ and $g \in X$ s.t. $|f| \le g \Longrightarrow f \in X$ and $\|f\|_X \le \|g\|_X$

- Köthe dual $X' := \{f \in L_0 : \int_\Omega f g \, d\mu$ exists $\forall g \in X\}$.

  – X' is always a closed subspace of the topological dual X^*

  – often $X' = X^*$, e.g. $L_p' = L_p^* = L_{p'}$ for $1 \le p < \infty$

- Generalized Hille-Tamarkin kernels

$$k \in X[X'] \, , \text{ i.e. } \quad \big\| \, \|k(x, \cdot)\|_{X'} \, \big\|_X < \infty \, ,$$

where X is any Banach function space.

- Let X be a Banach function space over $(\Omega, \mathcal{F}, \mu)$
  i.e. a Banach space $X \subset L_0(\Omega, \mathcal{F}, \mu)$ with the property:

  $f \in L_0$ and $g \in X$ s.t. $|f| \leq g \Longrightarrow f \in X$ and $\|f\|_X \leq \|g\|_X$

- Köthe dual $X' := \{f \in L_0 : \int_\Omega f g \, d\mu$ exists $\forall g \in X\}$.

  – X' is always a closed subspace of the topological dual X^*

  – often $X' = X^*$, e.g. $L_p' = L_p^* = L_{p'}$ for $1 \leq p < \infty$

- Generalized Hille-Tamarkin kernels

$$k \in X[X'], \text{ i.e. } \big\| \|k(x, \cdot)\|_{X'} \big\|_X < \infty,$$

where X is any Banach function space.

- $\curvearrowright$ factorization $T_k : X \xrightarrow{T_\ell} L_\infty \xrightarrow{M_g} X$ with

  $g(x) = \|k(x, \cdot)\|_{X'} \in X$, $\ell(x, t) = \frac{k(x,t)}{g(x)} \in L_\infty[X']$

- Weakly singular kernels on a bounded subset $\Omega \subset \mathbb{R}^d$:

$$k(x,t) = \frac{\ell(x,t)}{\|x-t\|^\alpha} \qquad \text{with } \ell \in L_\infty(\Omega \times \Omega) \text{ and } 0 < \alpha < d$$

- Weakly singular kernels on a bounded subset $\Omega \subset \mathbb{R}^d$:

$$k(x,t) = \frac{\ell(x,t)}{\|x-t\|^\alpha} \qquad \text{with } \ell \in L_\infty(\Omega \times \Omega) \text{ and } 0 < \alpha < d$$

- $p := d/(d-\alpha) \curvearrowright \forall x \in \Omega : \underbrace{\|k(x,\cdot)\|_{p',\infty}}_{\text{Lorentz norm}} \leq \|\ell\|_\infty$

- Weakly singular kernels on a bounded subset $\Omega \subset \mathbb{R}^d$:

$$k(x,t) = \frac{\ell(x,t)}{\|x-t\|^\alpha} \qquad \text{with } \ell \in L_\infty(\Omega\times\Omega) \text{ and } 0<\alpha<d$$

- $p := d/(d-\alpha) \ \curvearrowright\ \forall x \in \Omega : \ \underbrace{\|k(x,\cdot)\|_{p',\infty}}_{\text{Lorentz norm}} \leq \|\ell\|_\infty$

- eigenvalues are known: $(\lambda_n(T_k)) \in \begin{cases} \ell_{p,\infty} & \text{, if } p>2 \\ \ell_2 & \text{, if } p<2 \end{cases}$

  limiting case $p=2$: $\quad (\lambda_n(T_k)) \notin \ell_{2,\infty}$

- Weakly singular kernels on a bounded subset $\Omega \subset \mathbb{R}^d$:

$$k(x,t) = \frac{\ell(x,t)}{\|x-t\|^{\alpha}} \qquad \text{with } \ell \in L_\infty(\Omega \times \Omega) \text{ and } 0 < \alpha < d$$

- $p := d/(d-\alpha) \;\curvearrowright\; \forall x \in \Omega : \underbrace{\|k(x,\cdot)\|_{p',\infty}}_{\text{Lorentz norm}} \leq \|\ell\|_\infty$

- eigenvalues are known: $(\lambda_n(T_k)) \in \begin{cases} \ell_{p,\infty} & \text{, if } p > 2 \\ \ell_2 & \text{, if } p < 2 \end{cases}$

  limiting case $p = 2$: $\quad (\lambda_n(T_k)) \notin \ell_{2,\infty}$

- Note: $\quad k \in L_\infty[L_{p',\infty}] \subset L_{p,1}[L'_{p,1}]$

  $\curvearrowright$ Hille-Tamarkin type $X[X']$ with $X = L_{p,1}(\Omega)$

- Weakly singular kernels on a bounded subset $\Omega \subset \mathbb{R}^d$:

$$k(x,t) = \frac{\ell(x,t)}{\|x-t\|^\alpha} \qquad \text{with } \ell \in L_\infty(\Omega \times \Omega) \text{ and } 0 < \alpha < d$$

- $p := d/(d-\alpha) \;\curvearrowright\; \forall x \in \Omega : \underbrace{\|k(x,\cdot)\|_{p',\infty}}_{\text{Lorentz norm}} \leq \|\ell\|_\infty$

- eigenvalues are known: $(\lambda_n(T_k)) \in \begin{cases} \ell_{p,\infty} & \text{, if } p > 2 \\ \ell_2 & \text{, if } p < 2 \end{cases}$

  limiting case $p = 2$: $\quad (\lambda_n(T_k)) \notin \ell_{2,\infty}$

- Note: $\quad k \in L_\infty[L_{p',\infty}] \subset L_{p,1}[L'_{p,1}]$

  $\curvearrowright$ Hille-Tamarkin type $X[X']$ with $X = L_{p,1}(\Omega)$

- Our goal. Find (sharp) eigenvalue estimates for T_k with $k \in X[X']$.

  Question. Which geometric properties of X are responsible for the eigenvalue behaviour of such operators?

3. Geometry of Banach (function) spaces

3. Geometry of Banach (function) spaces

- A Banach space X has cotype p, $2 \le p < \infty$, if $\exists c > 0$ s.t.

$$\Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p} \le c\, \mathbb{E} \Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\|$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where ε_j are i.i.d. Bernoulli r.v.

3. Geometry of Banach (function) spaces

- A Banach space X has cotype p, $2 \le p < \infty$, if $\exists c > 0$ s.t.

$$\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p} \le c\,\mathbb{E}\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|$$

  for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where ε_j are i.i.d. Bernoulli r.v.

- A Banach function space X over (Ω, μ) is p-concave, $1 \le p < \infty$, if

$$\Big(\sum_{j=1}^{n} \|f_j\|^p\Big)^{1/p} \le c\,\Big\|\Big(\sum_{j=1}^{n} |f_j|^p\Big)^{1/p}\Big\|$$

3. Geometry of Banach (function) spaces

- A Banach space X has cotype p, $2 \le p < \infty$, if $\exists c > 0$ s.t.

$$\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p} \le c\,\mathbb{E}\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|$$

  for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where ε_j are i.i.d. Bernoulli r.v.

- A Banach function space X over (Ω, μ) is p-concave, $1 \le p < \infty$, if

$$\Big(\sum_{j=1}^{n} \|f_j\|^p\Big)^{1/p} \le c\,\Big\|\Big(\sum_{j=1}^{n} |f_j|^p\Big)^{1/p}\Big\|$$

- X Banach function space: X is 2-concave $\Longleftrightarrow X$ has cotype 2
  $2 < p < \infty$: p-concave $\Longrightarrow$ cotype $p \Longrightarrow (p+\varepsilon)$-concave $\forall \varepsilon > 0$

3. Geometry of Banach (function) spaces

- A Banach space X has cotype p, $2 \le p < \infty$, if $\exists c > 0$ s.t.

$$\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p} \le c\,\mathbb{E}\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|$$

  for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where ε_j are i.i.d. Bernoulli r.v.

- A Banach function space X over (Ω, μ) is p-concave, $1 \le p < \infty$, if

$$\Big(\sum_{j=1}^{n} \|f_j\|^p\Big)^{1/p} \le c\,\Big\|\Big(\sum_{j=1}^{n} |f_j|^p\Big)^{1/p}\Big\|$$

- X Banach function space: X is 2-concave $\Longleftrightarrow X$ has cotype 2
  $2 < p < \infty$: p-concave $\Longrightarrow$ cotype $p \Longrightarrow (p+\varepsilon)$-concave $\forall \varepsilon > 0$

- Examples. L_p is p-concave and has cotype $\max(p, 2)$
  cotype and concavity also known for $L_{p,q}$ and $L_p(\log L)_a$

4. Eigenvalue estimates

4. Eigenvalue estimates

- Notation:
  - X a Banach function space
  - $k \in X[X']$ a Hille-Tamarkin kernel
  - $T_k : X \to X$ the corresponding integral operator

4. Eigenvalue estimates

- Notation:
  – X a Banach function space
  – $k \in X[X']$ a Hille-Tamarkin kernel
  – $T_k : X \to X$ the corresponding integral operator

- Theorem 1. Let X be p-concave, $2 < p < \infty$.

  $\Longrightarrow T_k$ is a Riesz operator with eigenvalues in $\ell_{\max(p,2)}$.

4. Eigenvalue estimates

- Notation:
  - X a Banach function space
  - $k \in X[X']$ a Hille-Tamarkin kernel
  - $T_k : X \to X$ the corresponding integral operator

- Theorem 1. Let X be p-concave, $2 < p < \infty$.

  $\Longrightarrow T_k$ is a Riesz operator with eigenvalues in $\ell_{\max(p,2)}$.

- Theorem 2. Let X be of cotype p, $2 < p < \infty$.

  $\Longrightarrow T_k$ is a Riesz operator with eigenvalues in $\ell_{p,\infty}$.

4. Eigenvalue estimates

- Notation:
  - X a Banach function space
  - $k \in X[X']$ a Hille-Tamarkin kernel
  - $T_k : X \to X$ the corresponding integral operator

- Theorem 1. Let X be p-concave, $2 < p < \infty$.

  $\Longrightarrow T_k$ is a Riesz operator with eigenvalues in $\ell_{\max(p,2)}$.

- Theorem 2. Let X be of cotype p, $2 < p < \infty$.

  $\Longrightarrow T_k$ is a Riesz operator with eigenvalues in $\ell_{p,\infty}$.

- Remarks.
  - covers classical Hille-Tamarkin AND weakly singular kernels
  - extends this to much larger kernel classes
  - shows the interplay with geometry of the underlying spaces

Sketch of the proofs.

Sketch of the proofs.

For both theorems the same strategy works:

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p, 2)$-summing

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p, 2)$-summing

– by the ideal property, the same holds for T_k

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p, 2)$-summing

– by the ideal property, the same holds for T_k

– finally apply the classical eigenvalue results (König et al.)

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p, 2)$-summing

– by the ideal property, the same holds for T_k

– finally apply the classical eigenvalue results (König et al.)

- Limiting case. Let $2 \leq p < \infty$ and assume that X is NOT p-concave, but $(p + \varepsilon)$-concave $\forall \varepsilon > 0$.

  $\curvearrowright$ eigenvalues of T_k are in $\bigcap_{\varepsilon > 0} \ell_{p+\varepsilon}$

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p, 2)$-summing

– by the ideal property, the same holds for T_k

– finally apply the classical eigenvalue results (König et al.)

- Limiting case. Let $2 \leq p < \infty$ and assume that X is NOT p-concave, but $(p + \varepsilon)$-concave $\forall \varepsilon > 0$.
  $\curvearrowright$ eigenvalues of T_k are in $\bigcap_{\varepsilon>0} \ell_{p+\varepsilon}$

Question: Can one improve this?

Sketch of the proofs.

For both theorems the same strategy works:

– factorize T_k via a multiplication operator $M_g : L_\infty \to X$

– show that M_g is p-summing, resp. $(p,2)$-summing

– by the ideal property, the same holds for T_k

– finally apply the classical eigenvalue results (König et al.)

- Limiting case. Let $2 \le p < \infty$ and assume that X is NOT p-concave, but $(p+\varepsilon)$-concave $\forall \varepsilon > 0$.
  $\curvearrowright$ eigenvalues of T_k are in $\bigcap_{\varepsilon>0} \ell_{p+\varepsilon}$

Question: Can one improve this?

- Idea: One might expect that the

  p-concavity constants $M_{p,n}(X)$ of X evaluated with n vectors

  grow only moderately as $n \to \infty$, e.g. logarithmically.
  This is indeed the case in several examples!

- Theorem 3. Assume that $M_{p,n}(X) = \mathcal{O}(\log^a n)$ for some $2 \le p < \infty$ and $a > 0$, and let $k \in X[X']$. Then T_k is a Riesz operator in X and its eigenvalues satisfy

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^p \Big)^{1/p} \le c(1 + \log n)^a \|k\|_{X[X']} \,.$$

In particular, $\quad |\lambda_n(T_k)| = \mathcal{O}(n^{-1/p} \log^a n) \,.$

- Theorem 3. Assume that $M_{p,n}(X) = \mathcal{O}(\log^a n)$ for some $2 \le p < \infty$ and $a > 0$, and let $k \in X[X']$. Then T_k is a Riesz operator in X and its eigenvalues satisfy

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^p \Big)^{1/p} \le c(1 + \log n)^a \|k\|_{X[X']} .$$

In particular, $|\lambda_n(T_k)| = \mathcal{O}(n^{-1/p} \log^a n)$.

- Lemma (Lorentz and Zygmund spaces, defined over $[0, 1]$)
Let $1 \le q < 2 \le p < \infty$ and $a > 0$. Then we have

$$M_{2,n}\,(L_{2,q}) = \mathcal{O}\big((\log n)^{1/q-1/2}\big)\,,\; M_{p,n}\,(L_p(\log L)_a) = \mathcal{O}(\log^a n)\,.$$

- Theorem 3. Assume that $M_{p,n}(X) = \mathcal{O}(\log^a n)$ for some $2 \leq p < \infty$ and $a > 0$, and let $k \in X[X']$. Then T_k is a Riesz operator in X and its eigenvalues satisfy

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^p \Big)^{1/p} \leq c(1 + \log n)^a \|k\|_{X[X']} .$$

  In particular, $|\lambda_n(T_k)| = \mathcal{O}(n^{-1/p} \log^a n)$.

- Lemma (Lorentz and Zygmund spaces, defined over $[0, 1]$)
  Let $1 \leq q < 2 \leq p < \infty$ and $a > 0$. Then we have

$$M_{2,n}\left(L_{2,q}\right) = \mathcal{O}\Big((\log n)^{1/q-1/2}\Big) , \; M_{p,n}\left(L_p(\log L)_a\right) = \mathcal{O}(\log^a n) .$$

- combining the above results $\curvearrowright$ improved eigenvalue estimates

5. Some remarks on optimality

5. Some remarks on optimality

Example. It is well known that the Fourier series

$$\sum_{n=1}^{\infty} n^{-1/p}\, e^{2\pi i n x} \quad , \quad 1 < p < \infty$$

converges a.e. to a function $g \in L_{p',\infty}([0,1])$. Consider the integral operator T_k generated by the convolution kernel $k(x,t) = g(x-t)$.

5. Some remarks on optimality

Example. It is well known that the Fourier series

$$\sum_{n=1}^{\infty} n^{-1/p} \, e^{2\pi i n x} \quad , \quad 1 < p < \infty$$

converges a.e. to a function $g \in L_{p',\infty}([0,1])$. Consider the integral operator T_k generated by the convolution kernel $k(x,t) = g(x-t)$.

Then we have:

- $\lambda_n(T_k) = n^{-1/p}$ (eigenvalues of T_k = Fourier coefficients of g)
- k belongs to $X[X']$, where $X = L_{p,1}([0,1])$
- $L_{p,1}$ has cotype p, if $2 < p < \infty$

5. Some remarks on optimality

Example. It is well known that the Fourier series

$$\sum_{n=1}^{\infty} n^{-1/p} \, e^{2\pi i n x} \quad , \quad 1 < p < \infty$$

converges a.e. to a function $g \in L_{p',\infty}([0,1])$. Consider the integral operator T_k generated by the convolution kernel $k(x,t) = g(x-t)$.

Then we have:

– $\lambda_n(T_k) = n^{-1/p}$ (eigenvalues of T_k = Fourier coefficients of g)

– k belongs to $X[X']$, where $X = L_{p,1}([0,1])$

– $L_{p,1}$ has cotype p, if $2 < p < \infty$

$\curvearrowright$ optimality of Theorem 2 (for cotype p spaces)

5. Some remarks on optimality

Example. It is well known that the Fourier series

$$\sum_{n=1}^{\infty} n^{-1/p}\, e^{2\pi inx} \quad , \quad 1 < p < \infty$$

converges a.e. to a function $g \in L_{p',\infty}([0,1])$. Consider the integral operator T_k generated by the convolution kernel $k(x,t) = g(x-t)$.

Then we have:

– $\lambda_n(T_k) = n^{-1/p}$ (eigenvalues of T_k = Fourier coefficients of g)

– k belongs to $X[X']$, where $X = L_{p,1}([0,1])$

– $L_{p,1}$ has cotype p, if $2 < p < \infty$

$\curvearrowright$ optimality of Theorem 2 (for cotype p spaces)

Slightly more involved examples $\curvearrowright$ optimality of the remaining eigenvalue results mentioned in this talk.

Summary

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

– extension to more general kernel classes $X[X']$

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

– extension to more general kernel classes $X[X']$

– interplay eigenvalues of operators $\Longleftrightarrow$ geometry of spaces

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

– extension to more general kernel classes $X[X']$

– interplay eigenvalues of operators $\Longleftrightarrow$ geometry of spaces

– optimality for p-concave and cotype p-spaces

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

– extension to more general kernel classes $X[X']$

– interplay eigenvalues of operators $\Longleftrightarrow$ geometry of spaces

– optimality for p-concave and cotype p-spaces

– applications to Fourier analysis

Summary

– unified approach to Hille-Tamarkin and weakly singular operators

– extension to more general kernel classes $X[X']$

– interplay eigenvalues of operators $\Longleftrightarrow$ geometry of spaces

– optimality for p-concave and cotype p-spaces

– applications to Fourier analysis

THANK YOU FOR YOUR ATTENTION!