On reduced Sobolev imbeddings

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(based on joint papers with H.-J. Schmeisser)

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Plan of the talk

1 Introduction

Notation (definitions in a while)

$F^{s}_{p\,q}$ – the usual Lizorkin-Triebel space

$S^{\bar{r}}_{p\,q}F$ – the Lizorkin-Triebel space with dominating mixed derivatives

$S^{\bar{r},\overline{\alpha}}_{p\,q}F$ – the logarithmic (or refined) Lizorkin-Triebel space with dominating mixed derivatives

r instead of $\bar{r}$ if all components of $\bar{r}$ are equal, similarly for $\overline{\alpha}$

Our general concern:

Properties of Sobolev and more general spaces of weakly differentiable functions, when we neglect some of the derivatives

Specifically: We will be concerned with

- limiting imbeddings
- inequality of Gagliardo-Nirenberg type for Lizorkin-Triebel spaces with dominating mixed smoothness and logarithmic tuning,

2 Spaces and their reduced variants

$\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse, resp.,

$\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing C^∞ functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its dual, the space of tempered distributions

Let $\varphi_0 \in C^\infty(\mathbb{R}^N)$, $0 \le \varphi_0(x) \le 1$, $\varphi_0(x) = 1$ if $|x| \le 1$, and $\varphi_0(x) = 0$ if $|x| \ge 2$. Put

$$\varphi_1(x) = \varphi_0(x/2) - \varphi_0(x),$$
$$\varphi_j(x) = \varphi_1(2^{-j+1}x), \qquad j = 2, 3, \ldots .$$

Then $\sum_{j=0}^\infty \varphi_j(x) = 1$, $x \in \mathbb{R}^n$.

The system of functions $\{\varphi_j\}$ is the *smooth dyadic decomposition of the unity in* $\mathbb{R}^n$.

The usual *Lizorkin-Triebel space* $F^s_{p\,q}$:

Let $0 < p < \infty$, $0 < q \leq \infty$, $r \in \mathbb{R}$,

$$\|f|F^r_{p\,q}\| = \left\| \left(\sum_{j=0}^{\infty} \left| 2^{jr} \mathcal{F}^{-1} \left[\varphi_j \mathcal{F} f \right] (\,.\,) \right|^q \right)^{1/q} \middle| L_p \right\|$$

and

$$F^r_{p\,q}(\mathbb{R}^m \times \mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) : \ \|f|F^r_{p\,q}\| < \infty \}.$$

Connections with the “classical” spaces:

$$F^m_{p\,2}(\mathbb{R}^n) = W^m_p(\mathbb{R}^n) \qquad \text{if } 1 < p < \infty,\ m \in \mathbb{N}_0.$$

The *Lizorkin-Triebel space with dominating mixed derivatives and logarithmic tuning* (for the special splitting $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$):

$\{\varphi_j(x)\}_{j=0}^{\infty}$ and $\{\psi_j(x)\}_{j=0}^{\infty}$ be a smooth dyadic resolution of unity in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively

$0 < p < \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, r_2) \in \mathbb{R}^2$, $\overline{\alpha} = (\alpha_1, \alpha_2)$

We put

$$\|f|S^{\bar{r},\overline{\alpha}}_{p\,q}F\| = \Big\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Big| 2^{jr_1+kr_2}(1+j)^{\alpha_1}(1+k)^{\alpha_2} \mathcal{F}^{-1}\left[\varphi_j \otimes \psi_k \mathcal{F} f\right](\,.\,)\Big|^q \Big)^{1/q} \Big| L_p \Big\|$$

and

$$S^{\bar{r},\overline{\alpha}}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) :\ \|f|S^{\bar{r},\overline{\alpha}}_{p\,q}F\| < \infty\}.$$

If $\overline{\alpha} = (0,0)$, we get the usual Lizorkin-Triebel spaces with dominating mixed smoothness.

Note that

$$\begin{aligned} S^{\bar{s}}_{p\,2}F(\mathbb{R}^m \times \mathbb{R}^n) &= S^{\bar{s}}_{p}W(\mathbb{R}^m \times \mathbb{R}^n) \\ &= \{f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) : \\ &\qquad D^{\beta}_{x}D^{\gamma}_{y}f \in L_p(\mathbb{R}^m \times \mathbb{R}^n),\ |\beta| \leq s_1,\ |\gamma| \leq s_2\} \end{aligned}$$

if $\bar{s} = (s_1, s_2) \in \mathbb{N}^2_0$ and $1 < p < \infty$.

More general setting: p and q can be vectors.

References:

The Fourier analysis theory of $S^{\bar{s}}_{\bar{p}\,\bar{q}}F$ developed in 1980s (Schmeisser). The idea of such spaces goes back to Nikol'skii.

The logarithmic variant: Farkas and Leopold (2006), Triebel (2009, the case $\alpha_1 = \alpha_2$), Seyfried (2009, $1 < p < \infty$, $1 \leq q \leq \infty$). For $r_1, r_2 > 0$ the above spaces can be characterized by means of differences as spaces with logarithmic smoothness (Seyfried 2009).

Lizorkin and Nikol'skii (1990)—spaces with more general order of smoothness.

One classical prominent example of spaces with mixed derivatives—R. A. Adams 1986:

Let $M = (\alpha_{ij})$, $i = 1, \ldots, C(N,k)$, $j = 1, \ldots, N$, be a matrix whose entries are 0 or 1; given $1 \leq p < \infty$ assume that $\sum_{j=1}^{N} \alpha_{ij} = k$, $1 \leq k < N/p$, $C(N,k)$ is the number of all such rows (the respective combination number). Denote the rows of M by α_i.

Then

$$W_p^M = \{f \in L_p : \, D^{\alpha_i} f \in L_p, \; \alpha_i \text{ is a row of } M\}$$

is the *reduced Sobolev space*.

Adams proved that under the above assumptions

$$W_p^M \hookrightarrow L_q \qquad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{N}\,.$$

Further reductions are possible.

Suppose it is possible to choose from M a square submatrix such that all the sums over columns are the same and equal to k, i.e. to sums over the rows (the order of smoothness). Then the Sobolev imbeddings holds again ($1 \leq p < \infty$, $kp < N$).

It is tempting to call such matrices *magic squares*.

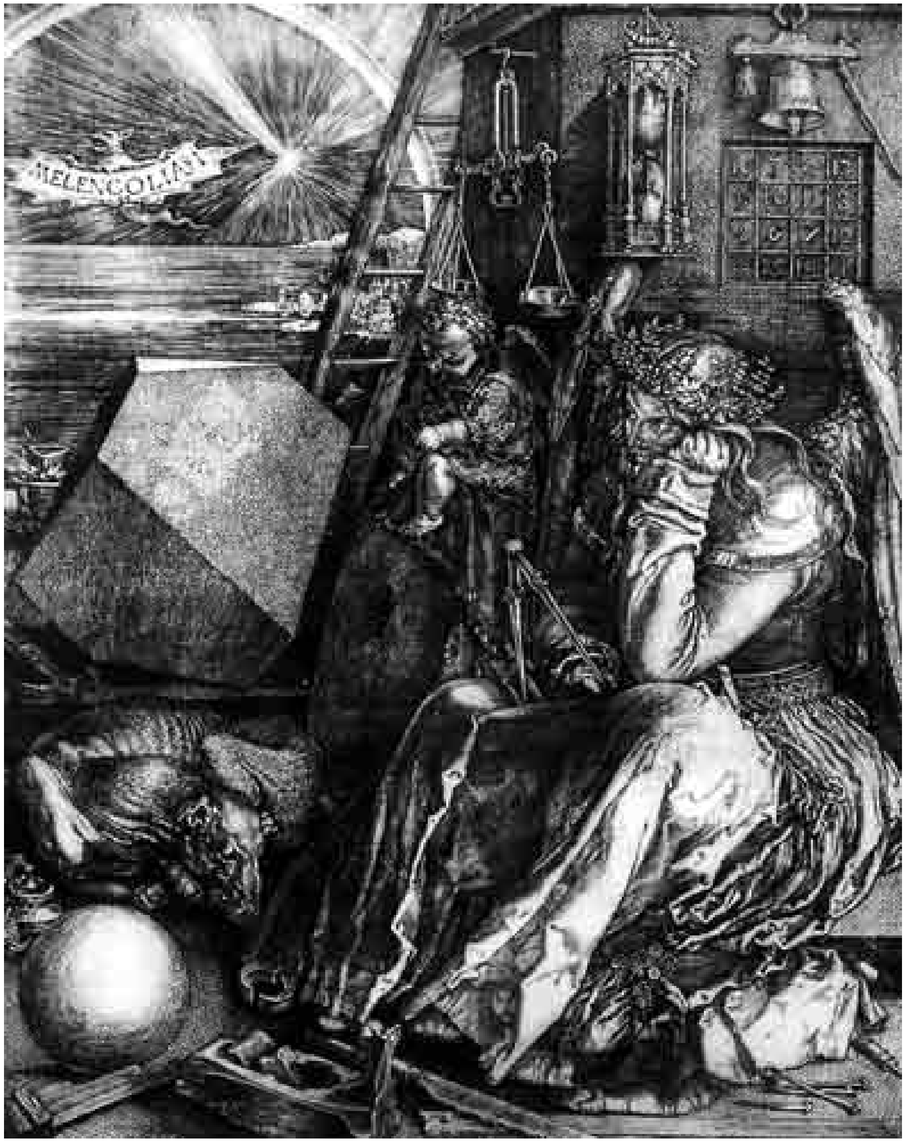
More generally, *magic squares* are square matrices, where sums over rows and columns are the same; possibly they have also some other puzzling properties. Historicians say that they attracted attention of curious people already in China, 3 000 years ago.

Being in Germany we make a short excursion into German Rennaisance.

Albrecht Dürer

* May 21, 1471, Imperial Free City of Nürnberg

† April 6, 1528, Nürnberg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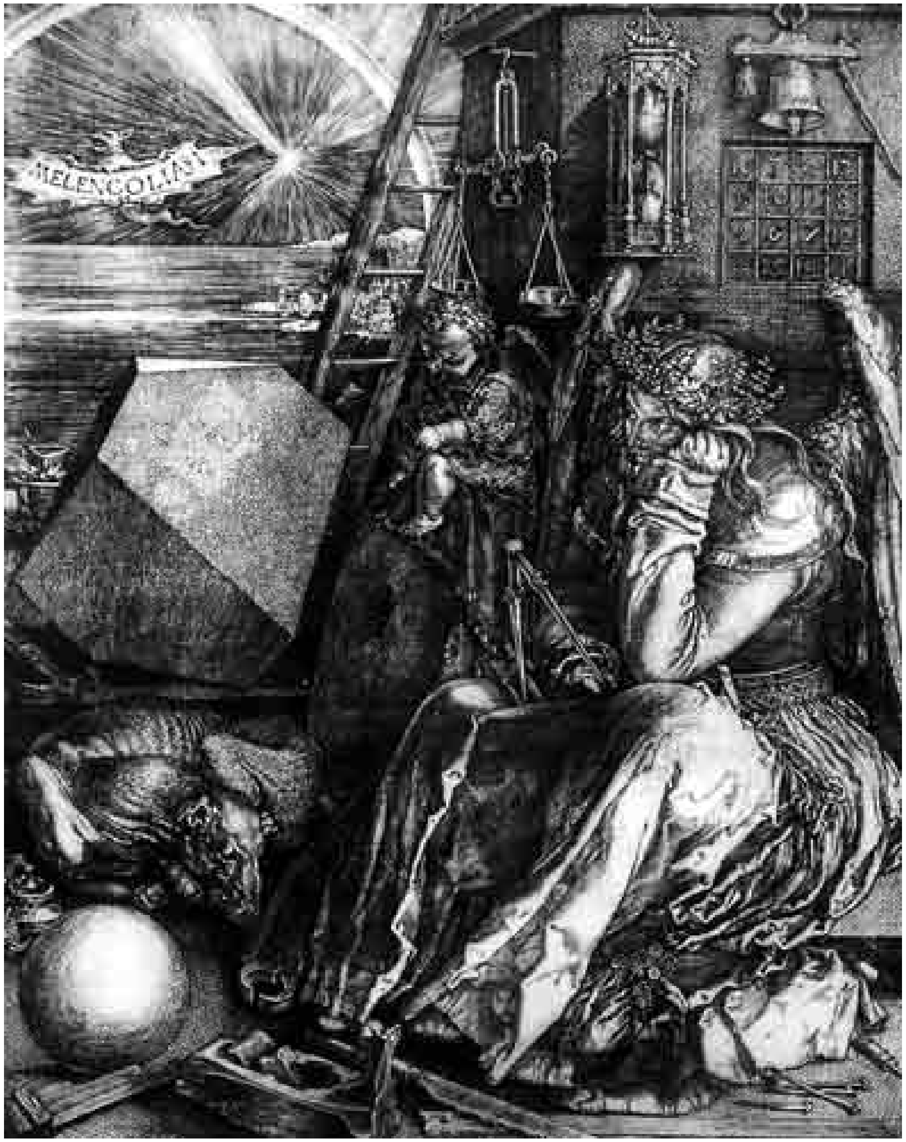
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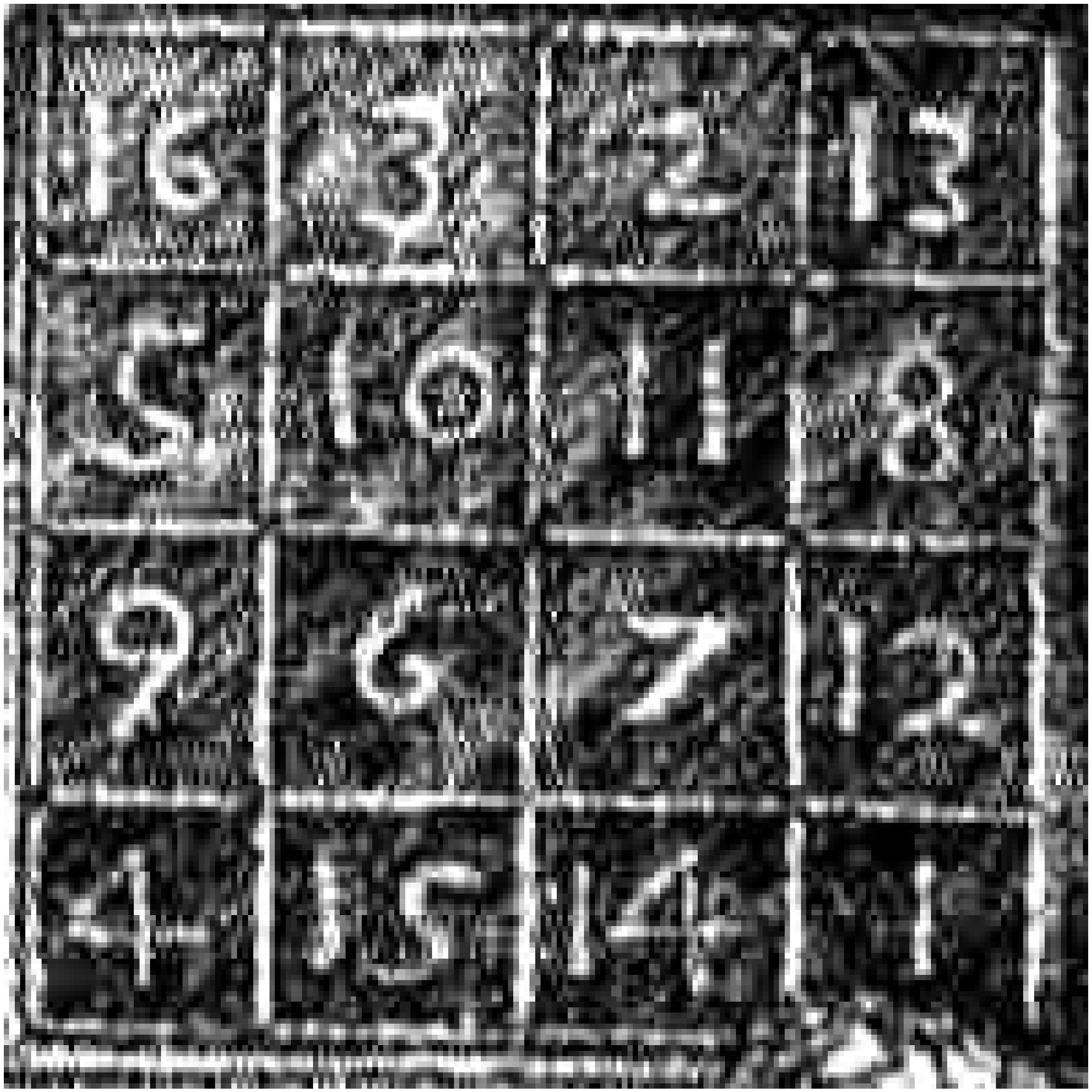

16 3 2 13
5 10 11 8
9 6 7 12
4 15 14 1

A real mathematics came much later: Birkhoff-von Neumann theorem on doubly stochastic matrices.

A square matrix is doubly stochastic iff it is a convex combination of permutation matrices. Permutation matrices are exactly all extremal points of the set of doubly stochastic matrices.

It is, however, not the end of possible reductions. If one can omit some rows in such a way that sums over columns are the same and the smoothness is a integer multiple of these sums, then the Sobolev imbedding holds again.

Probably there is no algebraic machinery behind it (representations for “parts” of doubly stochastic matrices) or it is well hidden in some abstract theories.

An imbedding theorem for very general spaces with generalized (Liouville) derivatives

Magaril-Il'yaev, 1986.

A scheme of using it will follow, demonstrated on an example.

3 Limiting imbeddings via Fourier analytic approach

Our interest in this section: contribution to limiting imbedding theory for spaces with dominating mixed smoothness. Results are scattered in various journals. Major references for joint papers with H.-J. Schmeisser:

Limiting imbeddings. The case of missing derivatives. Ricerche Mat. XLV(1996), 423-447.

Imbeddings of Brezis-Wainger type. The case of missing derivatives. Proc. Roy. Soc. Edinburgh 131 (2001), 667-700.

Refined limiting imbeddings for Sobolev spaces of vector-valued functions. J. Funct. Anal. 227 (2005), 372-388.

Critical imbeddings with multivariate rearrangements. Studia Math. 181(2007), 255-284.

Let us consider the archetypal case $N = 4$, $k = 2$, and

$$M = \{(1,1,0,0),(1,0,1,0),(0,1,0,1),(0,0,1,1)\}.$$

Let $f \in W_2^M(\mathbb{R}^4)$ be a C^∞ function supported in the unit ball.

Then, using just real analysis means, one can show that

$$\|f|L_q\| \leq cq\|f|W_2^M\|,$$

whereas in the isotropic case,

$$\|f|L_q\| \leq cq^{1/2}\|f|W_2^2\|,$$

Recall that such asymptotic estimates are the usual way how to prove that a function is exponentially integrable.

If f is a function $W^{k,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, $1 < p < \infty$, and $kp = N$, then f is in the Orlicz space $L_\Phi(\Omega)$ with $\Phi(t) = \exp t^{N/(N-k)} - 1$. The last fact can be expressed in terms of the asymptotic behaviour of L_q-norms of the function k; it reads here

$$\|u|L_q\| \leq cq^{1-k/N}\|f|W^{k,p}\|.$$

For unbounded domains one has to modify suitably Φ.

Let $\bar{p} = (p_1, \ldots, p_m)$, $\bar{q} = (q_1, \ldots, q_m)$, $\bar{r} = (r_1, \ldots, r_m)$ with $0 < p_j < \infty$, $0 < q_j \le \infty$, and $-\infty < r_j < \infty$, $j = 1, \ldots, m$, then

- (i)]

$$S^{\bar{r}}_{\bar{p},\bar{q}}B(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = \{f \in \mathcal{S}'(\mathbb{R}^N);$$
$$\|f|S^{\bar{r}}_{\bar{p},\bar{q}}B\| = \|2^{k_1 r_1 + \cdots + k_m r_m} f_{k_1,\ldots,k_m}(x)|L_{\bar{p}}|\ell_{\bar{q}}\| < \infty\},$$

(ii)

$$S^{\bar{r}}_{\bar{p},\bar{q}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = \{f \in \mathcal{S}'(\mathbb{R}^N);$$
$$\|f|S^{\bar{r}}_{\bar{p},\bar{q}}F\| = \|2^{k_1 r_1 + \cdots + k_m r_m} f_{k_1,\ldots,k_m}(x)|\ell_{\bar{p}}|L_{\bar{q}}\| < \infty\}.$$

3.1 Theorem. *Let* $1 < p_j < q_j < \infty$, $r_j = n_j/p_j \;\; (j = 1, \ldots, m)$. *If* $2 \le p_1 \le p_2 \le \cdots \le p_m$, *then there exists a constant* c *independent of* $\bar{q}$ *and* f *such that*

$$\|f|L_{\bar{q}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\| \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f|S^{\bar{r}}_{\bar{p},\bar{2}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\|$$

for all $f \in S^{\bar{r}}_{\bar{p},\bar{2}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$.

Let

$$f_j(\xi) := [\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)](\xi) \qquad j = 1, 2, \dots,$$

Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}^1$, $\alpha \in \mathbb{R}^1$, then

$$B_{pq}^{(s,\alpha)} = \Bigg\{ f \in \mathcal{S}' : \; \|f|B_{pq}^{(s,\alpha)}\|$$
$$= \left(\sum_{j=0}^{\infty} 2^{jsq} (j+1)^{-\alpha q} \|f_j(x)|L_p\|^q \right)^{1/q} < \infty \Bigg\}$$

(with the corresponding modification if $q = \infty$).

(Leopold 1998.)

Our interest: the spaces $B^{(1,\alpha)}_{\infty\infty}$, $\alpha > 0$, the *logarithmic Zygmund spaces*, defined by

$$B^{(1,\alpha)}_{\infty\infty} = \mathcal{C}^{(1,\alpha)} = \Bigg\{ f \in C : \; \|f|\mathcal{C}^{(1,\alpha)}\| \\ = \|f|L_\infty\| + \sup_{|h|\leq 1/2} \frac{\|\Delta_h^2 f(x)|L_\infty\|}{|h|\left[\log\left(1+\frac{1}{|h|}\right)\right]^\alpha} < \infty \Bigg\}$$

Lipschitzian counterpart of these spaces:

For $\alpha > 0$ we define the *logarithmic Lipschitz space* (*of order* α) $\mathrm{Lip}(1,\alpha)$ as

$$\mathrm{Lip}(1,\alpha) = \Bigg\{ f \in C : \ \|f|\,\mathrm{Lip}(1,\alpha)\| \\ = \|f|L_\infty\| + \sup_{|h|<1/2} \frac{\|\Delta_h f(x)|L_\infty\|}{|h|\left[\log\left(1+\frac{1}{|h|}\right)\right]^\alpha} < \infty \Bigg\}$$

For $f \in \mathrm{Lip}(1,\alpha)$ we also talk about the *almost Lipschitz continuous functions* if there is no need to specify the particular value of α.

$W_p^{N/p+1}(\Omega)$ ($\partial\Omega$ sufficiently regular) is imbedded into any Hölder space $\mathcal{C}^\alpha(\Omega)$ with $0 < \alpha < 1$. On the other hand, functions in $W_p^{N/p+1}(\Omega)$ need not be Lipschitz continuous.

Brézis and Wainger:

$$|f(x) - f(y)| \leq C|x-y||\log|x-y||^{1-1/p}\|f|W_p^{N/p+1}\|, \quad x, y \in \mathbb{R}^N,\ |x-y| < 1/2, \tag{3.1}$$

for functions in $W_p^{N/p+1}$, $1 < p < \infty$.

We shall survey some of the results.

A rule of thumb: Neglecting a proper subset of derivatives in the sublimiting situation does not affect the Sobolev imbedding, however, norms of imbeddings increase, which in turn gives a worse exponential integrability in the limiting case.

Sickel and Triebel 1995:

$$B^{N/p+1}_{p\,q} \hookrightarrow B^1_{\infty\,q},$$
$$F^{N/p+1}_{p\,q} \hookrightarrow B^1_{\infty\,p},$$

and also

$$B^1_{\infty\,q} \hookrightarrow C^1 \quad \text{(or } \mathrm{Lip}\,1\text{)}$$

if and only if $0 < q \leq 1$.

3.2 Theorem. *The following imbeddings hold:*

(i) *If* $1 < q \le \infty$, *then*

$$B^1_{\infty\, q} \hookrightarrow \mathrm{Lip}(1, 1/q').$$

(ii) *If* $0 < \alpha < \infty$, *then*

$$B^{(1,\alpha)}_{\infty 1} \hookrightarrow \mathrm{Lip}(1, \alpha).$$

In our terminology the imbedding theorem due to Brézis and Wainger reads

$$F^{N/p+1}_{p\,2} \hookrightarrow \mathrm{Lip}(1, 1/p').$$

Further definitions—we shall restrict ourselves to the special splitting $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

We put $\bar{p} = (p_1, p_2)$, $\bar{1} = (1,1)$, $\bar{2} = (2,2)$, $\bar{\infty} = (\infty, \infty)$, $\bar{0} = (0,0)$, $x = (x^1, x^2)$ with $x^1 \in \mathbb{R}^{n_1}$ and $x^2 \in \mathbb{R}^{n_2}$, etc. Let $\{\varphi(\xi^1)\}_{j=0}^{\infty}$ and $\{\psi(\xi^1)\}_{j=0}^{\infty}$ be a smooth dyadic resolution of unity in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. For $f \in \mathcal{S}'(\mathbb{R}^N)$ we put

$$f_{j,k}(x^1, x^2) := [\mathcal{F}^{-1} \varphi_j(\xi^1) \psi_k(\xi^2) \mathcal{F} f](x^1, x^2)$$

$(j, k = 0, 1, \ldots)$.

(i) Let $\bar{p} = (p_1, p_2)$, $\bar{q} = (q_1, q_2)$, $\bar{r} = (r_1, r_2)$ with $0 < p_i, q_i, \leq \infty$, $r_i \in \mathbb{R}^1$ $(i = 1, 2)$. Then

$$S^{\bar{r}}_{\bar{p}\,\bar{q}}B(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^N) :\ \|f|S^{\bar{r}}_{\bar{p}\,\bar{q}}B\| = \|2^{r_1 j + r_2 k} f_{j,k}(x^1, x^2)|L_{p_1|x^1}|L_{p_2|x^2}|\ell_{q_1|j}|\ell_{q_2|k}\| < \infty\}$$

and

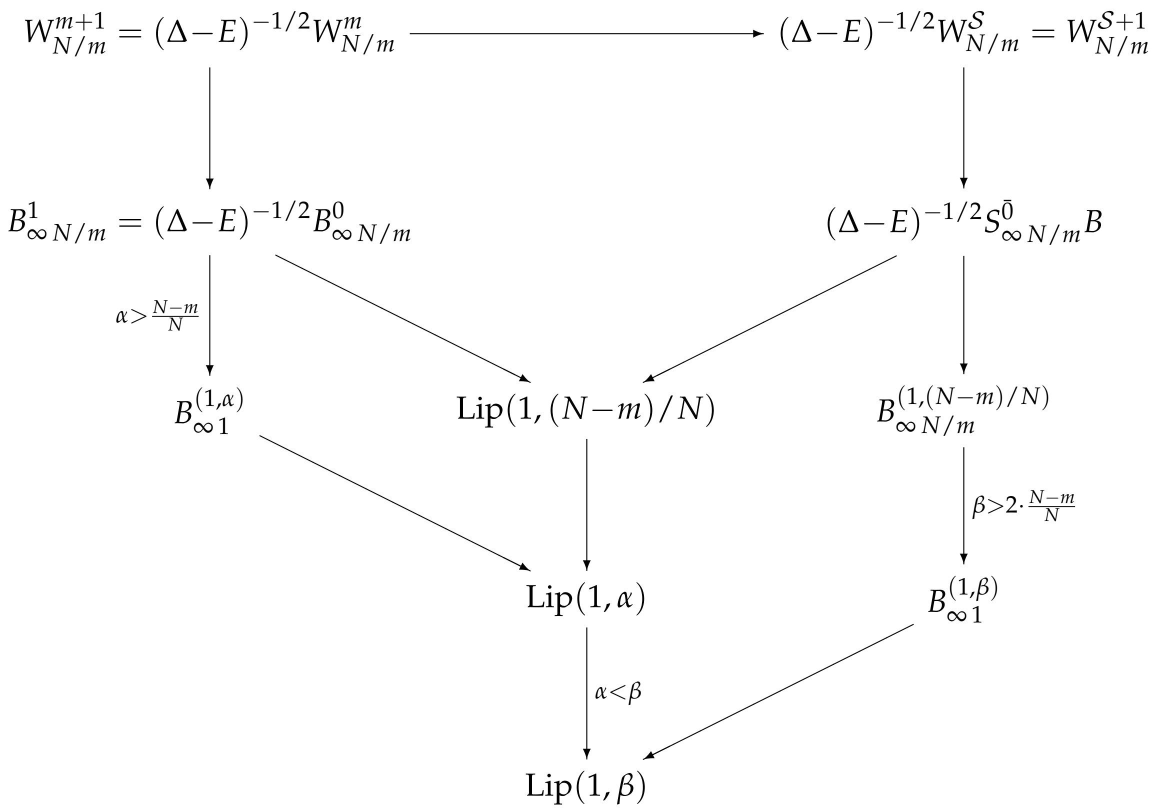
$$SB^{\bar{r}}_{\bar{p}\,\bar{q}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^N) :\ \|f|SB^{\bar{r}}_{\bar{p}\,\bar{q}}\| = \|2^{r_1 j + r_2 k} f_{j,k}(x^1, x^2)|L_{p_1|x^1}|\ell_{q_1|j}|L_{p_2|x^2}||\ell_{q_2|k}\| < \infty\}.$$

(ii) Let additionally $0 < p_i < \infty$ $(i = 1, 2)$. Then

$$S^{\bar{r}}_{\bar{p}\,\bar{q}}F(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^N) :\ \|f|S^{\bar{r}}_{\bar{p}\,\bar{q}}F\| = \|2^{r_1 j + r_2 k} f_{j,k}(x^1, x^2)|\ell_{q_1|j}|\ell_{q_2|k}|L_{p_1|x^1}|L_{p_2|x^2}\| < \infty\}.$$

The previous claims can be generalized to mixed norm setup.

A diagram for the BW-imbeddings



4 Gagliardo-Nirenberg inequalities

Natural interest in other relevant relations between the norms.

Brezis and Mironescu proved in 2001 that

$$\|f|F^{s}_{p\,q}(\mathbb{R}^n)\| \leq \|f|F^{s_1}_{p_1\,q_1}(\mathbb{R}^n)\|^{\theta}\,\|f|F^{s_2}_{p_2\,q_2}(\mathbb{R}^n)\|^{1-\theta} \qquad (4.1)$$

provided that $0 < q, q_1, q_2 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, and $s = \theta s_1 + (1-\theta)s_2$.

If we pass to spaces with dominating mixed smoothness $S^r_{p\,q}F(\mathbb{R}^n)$, the situation unfortunately and also rather surprisingly changes.

Hansen proved in 2009 that

$$\|f|S^{s}_{p\,q}F(\mathbb{R}^n)\| \le c\|f|S^{s_1}_{p_1\,q_1}F(\mathbb{R}^n)\|^{\theta}\,\|f|S^{s_2}_{p_2\,q_2}F(\mathbb{R}^n)\|^{1-\theta} \tag{4.2}$$

iff

$$0<\theta<1, \quad \frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}, \quad s=\theta s_1+(1-\theta)s_2 \tag{4.3}$$

$$\text{and} \quad \frac{1}{q}\le\frac{\theta}{q_1}+\frac{1-\theta}{q_2}.$$

The problem is what happens with inequalities of type (4.2) for refined spaces with dominating mixed smoothness without the last condition in (4.3).

First let us observe that inequality of type (4.2) can be proved just with help of Hölder's inequality provided equality is supposed in the formula for q.

4.1 Proposition. *Assume that* $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$, $\bar{u}, \bar{v} \in \mathbb{R}^2$, $0 < \theta < 1$, $\overline{\beta}, \overline{\gamma} \in \mathbb{R}^2$. *Let*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

and

$$\bar{r} = \theta\bar{u} + (1-\theta)\bar{v} \quad \overline{\alpha} = \theta\overline{\beta} + (1-\theta)\overline{\gamma}.$$

Then

$$\|f|S^{\bar{r},\overline{\alpha}}_{p\,q}F(\mathbb{R}^n)\| \le c\|f|S^{\bar{u},\overline{\beta}}_{p_1 q_1}F(\mathbb{R}^n)\|^{\theta}\,\|f|S^{\bar{v},\overline{\gamma}}_{p_2 q_2}F(\mathbb{R}^n)\|^{1-\theta}. \tag{4.4}$$

Proof. Let us write

$$f_{jk}(x) = \mathcal{F}^{-1}[\phi_j \otimes \psi_k \mathcal{F} f](x), \qquad x \in \mathbb{R}^m \times \mathbb{R}^n,\ (j,k) \in \mathbb{N}_0^2,$$

and

$$\begin{aligned}
|a_{jk}(x)| &= \left|2^{jr_1+kr_2}(1+j)^{\alpha_1}(1+k)^{\alpha_2} f_{jk}(x)\right| \\
&= \left|2^{ju_1+ku_2}(1+j)^{\beta_1}(1+k)^{\beta_2} f_{jk}(x)\right|^{\theta} \\
&\qquad \left|2^{jv_1+kv_2}(1+j)^{\gamma_1}(1+k)^{\gamma_2} f_{jk}(x)\right|^{1-\theta} \\
&= |b_{jk}(x)|^{\theta} |c_{jk}(x)|^{1-\theta}.
\end{aligned}$$

Then the claim follows by repeating application of Hölder's inequality with $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$.

Indeed,

$$
\begin{aligned}
&\Big\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}(\,.\,)|^q \Big)^{1/q} \Big| L_p \Big\| \\
&= \Big\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |b_{jk}(\,.\,)|^{\theta q} |c_{jk}(\,.\,)|^{(1-\theta)q} \Big)^{1/q} \Big| L_p \Big\| \\
&\le \Big\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |b_{jk}(\,.\,)|^{q_1} \Big)^{\theta/q_1} \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |c_{jk}(\,.\,)|^{q_2} \Big)^{(1-\theta)/q_2} \Big| L_p \Big\| \\
&\le \| \{b_{jk}(\,.\,)\} | L_{p_1}(\ell_{q_1}) \|^{\theta} \| \{c_{jk}(\,.\,)\} | L_{p_2}(\ell_{q_2}) \|^{1-\theta}.
\end{aligned}
$$

□

We get two consequences:

Choosing $\overline{\beta} = 0$ we obtain

$$\|f|S^{\bar{u},\overline{\alpha}}_{p\,q}F\| \le c\,\|f|S^{\bar{u}}_{p\,\infty}F\| \qquad \text{if} \quad \alpha < -\frac{1}{q}\,.$$

Further,

$$\|f|S^{\bar{r},\overline{\alpha}}_{p\,q}F(\mathbb{R}^n)\| \le c\|f|S^{\bar{u}}_{p_1\,\infty}F(\mathbb{R}^n)\|^{\theta}\,\|f|S^{\bar{v}}_{p_2\,\infty}F(\mathbb{R}^n)\|^{1-\theta} \tag{4.5}$$

holds for $0 < q \le \infty$, $\overline{\alpha} = (\alpha_1, \alpha_2)$ with $\alpha_i < -1/q$ ($i = 1, 2$), and

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \bar{r} = \theta\bar{u} + (1-\theta)\bar{v},$$

which is inequality of type (4.1) for spaces with dominating mixed smoothness, where we pay some logarithmic price.

Let now $u_1 = u_2 = u$ and $v_1 = v_2 = v$. Then the above estimate (4.5) can be improved with respect to α.

4.2 Theorem. *Let* $0 < p_1, p_2 < \infty$ *(*$p_1 \neq p_2$*),* $0 < q < \infty$*,* $u, v \in \mathbb{R}$*, and let*

$$r = \theta u + (1-\theta)v, \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then

$$\|f|S^{\bar{r},-1/(2q)}_{p\,q}F(\mathbb{R}^n)\| \le c\|f|S^{u}_{p_1\,\infty}F(\mathbb{R}^n)\|^{\theta}\,\|f|S^{v}_{p_2\,\infty}F(\mathbb{R}^n)\|^{1-\theta} \tag{4.6}$$

for all $f \in S^{u}_{p_1\,\infty}F(\mathbb{R}^m \times \mathbb{R}^n) \cap S^{v}_{p_2\,\infty}F(\mathbb{R}^m \times \mathbb{R}^n)$.

The proof cannot be reproduced here. The first main ingredience is the following remarkable estimate due to Brezis and Mironescu:

4.3 Proposition. *Let* $0 < \theta < 1$, $0 < q < \infty$, *and* $r = \theta u + (1-\theta)v$. *Then there exists* $c > 0$ *such that*

$$\left\|\{2^{\ell r} d_\ell\}|\ell_q\right\| \leq c \left\|\{2^{\ell u} d_\ell\}|\ell_\infty\right\|^{\theta} \left\|\{2^{\ell v} d_\ell\}|\ell_\infty\right\|^{1-\theta}.$$

The second key point is inequality, expressing a fine balance for diagonal sums:

$$\sum_{j+k=\ell} [(1+j)(1+k)]^{-1/2} \sim 1$$

with equivalence independent of $\ell \in \mathbb{N}$.

4.4 Remark. We can replace $f_{jk}(x)$ by $(1+j)^\alpha(1+k)^\alpha f_{jk}(x)$ with arbitrary $\alpha \in \mathbb{R}$ in the above considerations. Hence under the assumptions of Theorem 4.2 we get

$$\left\| f|S^{r,\alpha-1/(2q)}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)\right\| \\ \leq c \left\| f|S^{u,\alpha}_{p_1\infty}F(\mathbb{R}^m \times \mathbb{R}^n)\right\|^\theta \left\| f|S^{v,\alpha}_{p_2\infty}F(\mathbb{R}^m \times \mathbb{R}^n)\right\|^{1-\theta}$$

for all $f \in S^{u,\alpha}_{p_1\infty}F(\mathbb{R}^m \times \mathbb{R}^n) \cap S^{v,\alpha}_{p_2\infty}F(\mathbb{R}^m \times \mathbb{R}^n)$ and arbitrary real α.

No relevant results of this sort for the reduced spaces (except the easy case $p = 1$).