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Global compensated compactness theorem for general differential operators of first order

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1. Introduction

Div-Curl Lemma. $D \subset \mathbb{R}^n$: any open set

(i) $$u_m \rightharpoonup u, \quad v_m \rightharpoonup v \quad \text{weakly in } L^2(D).$$

(ii) $\{\text{div } u_m\}_{m=1}^{\infty}$ and $\{\text{rot } v_m\}_{m=1}^{\infty}$ are bounded in $L^2(D)$.

$\Longrightarrow$

$$u_m \cdot v_m \rightharpoonup u \cdot v \quad \text{in the sense of distributons in } D,$$

i,e.,

$$\int_D \varphi u_m \cdot v_m dx \to \int_D \varphi u \cdot v dx \quad \text{for } \forall \varphi \in C_0^\infty(D),$$

where $u \cdot v \equiv \sum_{k=1}^{n} u^k v^k$ for $u = (u^1, \cdots, u^n)$ and $v = (v^1, \cdots, v^n)$.

c.f. *Rellich's compact theorem.*

$\Omega \subset \mathbb{R}^n$: bounded domain

(i)
$$u_m \rightharpoonup u, \quad v_m \rightharpoonup v \quad \text{weakly in } L^2(\Omega).$$

(ii)' $\{\nabla u_m\}_{m=1}^{\infty}$ and $\{\nabla v_m\}_{m=1}^{\infty}$ are bounded in $L^2(\Omega)$, i.e.,

$$\left\{\frac{\partial u_m^k}{\partial x^l}\right\}_{m=1}^{\infty} \text{ and } \left\{\frac{\partial v_m^k}{\partial x^l}\right\}_{m=1}^{\infty} \text{ are bounded in } L^2(\Omega) \text{ for } \forall k, l = 1, \cdots, n$$

$\Longrightarrow$

$$u_m \to u, \quad v_m \to v \quad \text{strongly in } L^2(\Omega),$$

which yields

$$u_m^k v_m^l \to u^k v^l \quad \text{strongly in } L^1(\Omega), \quad \forall k, l = 1, \cdots, n.$$

Question.

$\Omega \subset \mathbb{R}^n$: bounded domain

(i) $$u_m \rightharpoonup u, \quad v_m \rightharpoonup v \quad \text{weakly in } L^2(\Omega).$$

(ii) $\{A_1(\cdot, D)u_m\}_{m=1}^{\infty}$ and $\{A_2(\cdot, D)v_m\}_{m=1}^{\infty}$ are bounded in $L^2(\Omega)$

$\Longrightarrow$

(*) $$u_m \cdot v_m \to u \cdot v \quad \text{strongly in } L^1(\Omega) \ ?$$

Find a condition on the differential operators $A_1(x, D)$ and $A_2(x, D)$ so that (*) holds.

Example . $A_1(x, D) = \text{div}$, $A_2(x, D) = \text{rot}$

Formulation. $\Omega \subset \mathbb{R}^n$: bounded domain with $\partial\Omega \in C^\infty$.
$A(x, D) : C^\infty(\bar{\Omega})^l \mapsto C^\infty(\bar{\Omega})^d$

$$\begin{aligned} A(x,D)u &= {}^t\left(\sum_{j=1}^{l} A_{1j}(x,D)u_j, \cdots, \sum_{j=1}^{l} A_{dj}(x,D)u_j\right), \\ u &= {}^t(u_1, \cdots, u_l) \in C^\infty(\bar{\Omega})^l, \end{aligned}$$

$$A_{ij}(x,D) = \sum_{k=1}^{n} a_{ijk}(x)\frac{\partial}{\partial x_k} + b_{ij}(x), \quad x \in \bar{\Omega}, D = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$$

with $a_{ijk}, b_{ij} \in C^\infty(\bar{\Omega})$ for $i = 1, \cdots, d$, $j = 1, \cdots, l$, $k = 1, \cdots, n$.

$\exists A'(x, D) : C^\infty(\bar{\Omega})^d \mapsto C^\infty(\bar{\Omega})^l$,
$\exists B(x, \nu) : C^\infty(\bar{\Omega})^l \mapsto C^\infty(\partial\Omega)^d$, $\quad \exists B'(x, \nu) : C^\infty(\bar{\Omega})^d \mapsto C^\infty(\partial\Omega)^l$

such that

(1) $\qquad (A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle B(\cdot, \nu)u, \varphi\rangle_{\partial\Omega}$,

(2) $\qquad (A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle u, B'(\cdot, \nu)\varphi\rangle_{\partial\Omega}$

for $\forall u \in C^\infty(\bar{\Omega})^l$, $\forall \varphi \in C^\infty(\bar{\Omega})^d$,
$\nu = (\nu_1, \cdots, \nu_n)$: unit outer normal to $\partial\Omega$

Remark 1.

$$
\begin{aligned}
B(x,\nu)u &= \left(\sum_{j=1}^{l} B_{1j}(x,\nu)u_j, \cdots, \sum_{j=1}^{l} B_{dj}(x,\nu)u_j\right), \\
u &= {}^t(u_1,\cdots,u_l) \in C^\infty(\bar{\Omega})^l, \\
B_{ij}(x,\nu) &= \sum_{k=1}^{n} a_{ijk}(x)\nu_k, \quad i=1,\cdots,d, j=1,\cdots,l, \\
A'(x,D)\varphi &= {}^t\left(\sum_{i=1}^{d} A'_{1i}(x,D)\varphi_i, \cdots, \sum_{i=1}^{d} A'_{li}(x,D)\varphi_i\right) \\
A'_{ji}(x,D) &= -\sum_{k=1}^{n} a_{ijk}(x)\frac{\partial}{\partial x_k} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} a_{ijk}(x) + b_{ij}(x), \\
B'(x,\nu)\varphi &= \left(\sum_{i=1}^{d} B'_{1i}(x,\nu)\varphi_i, \cdots, \sum_{i=1}^{d} B'_{li}(x,\nu)\varphi_i\right), \\
B'_{ji}(x,\nu) &= \sum_{k=1}^{n} a_{ijk}(x)\nu_k, \quad j=1,\cdots,l, i=1,\cdots,d, \\
\varphi &= {}^t(\varphi_1,\cdots,\varphi_d) \in C^\infty(\bar{\Omega})^d,
\end{aligned}
$$

Remark 2.(generalized Stokes formula) (1)

$u \in L^2(\Omega)^l$, $A(x,D)u \in L^2(\Omega)^d \Longrightarrow B(x,\nu)u \in H^{-\frac{1}{2}}(\partial\Omega)^d \equiv (H^{\frac{1}{2}}(\partial\Omega)^d)^*$,

$$(A(\cdot,D)u,\varphi) - (u, A'(\cdot,D)\varphi) = \langle B(\cdot,\nu)u,\varphi\rangle_{\partial\Omega}, \quad \forall \varphi \in H^1(\Omega)^d,$$

(2)

$\varphi \in L^2(\Omega)^d$, $A'(x,D)\varphi \in L^2(\Omega)^l \Longrightarrow B'(x,\nu)\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^l \equiv (H^{\frac{1}{2}}(\partial\Omega)^l)^*$,

$$(A(\cdot,D)u,\varphi) - (u, A'(\cdot,D)\varphi) = \langle u, B'(\cdot,\nu)\varphi\rangle_{\partial\Omega}, \quad \forall u \in H^1(\Omega)^l,$$

Consider

$\{A_\alpha(x,D), A'_\alpha(x,D), B_\alpha(x,\nu), B'_\alpha(x,\nu)\}_{\alpha=1,2}$ with $l_1 = l_2 = l$, i.e.,

$$A_1(x,D) : H^1(\Omega)^l \mapsto L^2(\Omega)^{d_1}, \quad A_2(x,D) : H^1(\Omega)^l \mapsto L^2(\Omega)^{d_2}$$

Assumption.

$$\begin{aligned}
\|\nabla u\| &\leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)}), \\
\|\nabla u\| &\leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_2 u\|_{H^{\frac{1}{2}}(\partial\Omega)})
\end{aligned}$$

for $\forall u \in H^1(\Omega)^l$. $\|\cdot\|$: L^2-norm on Ω.

Theorem. *Let two pairs* $\{A_\alpha(x,D), A'_\alpha(x,D), B_\alpha(x,\nu), B'_\alpha(x,\nu)\}_{\alpha=1,2}$ *satisfy the generalized Stokes integral formula. Let the Assumption hold. We assume the cancellation property*

$$A_2 A'_1 = 0, \quad A_1 A'_2 = 0. \tag{3}$$

Suppose that $\{u_m\}_{m=1}^\infty$ *and* $\{v_m\}_{m=1}^\infty$ *satisfy*

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^l$;

(ii) $\{A_1 u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_1}$ *and* $\{A_2 v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_2}$;

(iii) *Either* $\{B_1 u_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_1}$ *or* $\{B_2 v_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_2}$.

$\Longrightarrow$

$$\int_\Omega u_m(x)\cdot v_m(x)dx \to \int_\Omega u(x)\cdot v(x)dx \quad \text{as } m\to\infty.$$

Remark.

$$A_\alpha(x,D)u = {}^t\left(\sum_{j=1}^{l} A^{(\alpha)}_{1j}(x,D)u_j, \cdots, \sum_{j=1}^{l} A^{(\alpha)}_{dj}(x,D)u_j\right)$$

$$A^{(\alpha)}_{ij}(x,D) = \sum_{k=1}^{n} a^{(\alpha)}_{ijk}(x)\frac{\partial}{\partial x_k} + b^{(\alpha)}_{ij}(x), i = 1, \cdots, d_\alpha, j = 1, \cdots, l, \alpha = 1, 2,$$

the cancellation property (3) $\Longleftrightarrow$

$$\sum_{j=1}^{l} (a^{(\alpha)}_{rjs}a^{(\beta)}_{ijk} + a^{(\alpha)}_{rjk}a^{(\beta)}_{ijs}) = 0,$$

$$\alpha, \beta = 1, 2, \quad \alpha \neq \beta, \quad 1 \leqq \forall s, \forall k \leqq n, \quad r = 1, \cdots, d_\alpha, i = 1, \cdots d_\beta,$$

$$\sum_{j=1}^{l}\left(\sum_{\mu=1}^{n} a^{(\alpha)}_{rj\mu}\frac{\partial a^{(\beta)}_{ijk}}{\partial x_\mu} + a^{(\alpha)}_{rjk}\sum_{\mu=1}^{n}\frac{\partial a^{(\beta)}_{ij\mu}}{\partial x_\mu} - a^{(\alpha)}_{rjk}b^{(\beta)}_{ij} - a^{(\beta)}_{ijk}b^{(\alpha)}_{rj}\right) = 0,$$

$$\alpha, \beta = 1, 2, \quad \alpha \neq \beta, \quad 1 \leqq \forall k \leqq n, \quad r = 1, \cdots, d_\alpha, i = 1, \cdots d_\beta,$$

$$\sum_{j=1}^{l}\left(\sum_{\mu=1}^{n} a^{(\alpha)}_{rj\mu}(-\sum_{\sigma=1}^{n}\frac{\partial^2 a^{(\beta)}_{ij\sigma}}{\partial x_\mu \partial x_\sigma} + \frac{\partial b^{(\beta)}_{ij}}{\partial x_\mu}) + b^{(\alpha)}_{rj}(-\sum_{\mu=1}^{n}\frac{\partial a^{(\beta)}_{ij\mu}}{\partial x_\mu} + b^{(\beta)}_{ij})\right) = 0,$$

$$\alpha, \beta = 1, 2, \quad \alpha \neq \beta, \quad r = 1, \cdots, d_\alpha, i = 1, \cdots d_\beta.$$

2. Applications.

2.1. Global Div-Curl lemma in bounded domains

Corollary 1 $\Omega \subset \mathbb{R}^3$: *bounded domain with* $\partial\Omega \in C^\infty$. *Suppose that* $\{u_m\}_{m=1}^\infty$ *and* $\{v_m\}_{m=1}^\infty$ *are sequences of* 3-*D vector fields in* Ω *satisfying*

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^3$;

(ii) $\{\text{div } u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)$, *and*
$\{\text{rot } v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^3$;

(iii) *Either*
$\{u_m \cdot \nu\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$, *or*
$\{v_m \times \nu\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^3$

$\Longrightarrow$

$$\int_\Omega u_m(x) \cdot v_m(x)dx \to \int_\Omega u(x) \cdot v(x)dx \quad \text{as } m \to \infty.$$

Corollary 2 $\Omega \subset \mathbb{R}^3$: *bounded domain with* $\partial\Omega \in C^\infty$.
Let $b = {}^t(b_1, b_2, b_3) \in C^1(\bar{\Omega})^3$ *be as* $\mathrm{rot}\ b = 0$.
Suppose that $\{u_m\}_{m=1}^\infty$ *and* $\{v_m\}_{m=1}^\infty$ *are sequences of 3-D vector fields in* Ω *satisfying*

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^3$;

(ii) $\{\mathrm{div}\ u_m + b \cdot u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)$, *and*
$\{\mathrm{rot}\ v_m + b \times v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^3$;

(iii) *Either*
$\{u_m \cdot \nu\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$, *or*
$\{v_m \times \nu\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^3$.

$\Longrightarrow$

$$\int_\Omega u_m(x) \cdot v_m(x)dx \to \int_\Omega u(x) \cdot v(x)dx \quad \text{as } m \to \infty.$$

Corollary 3 $\Omega \subset \mathbb{R}^n (n \geq 2)$:*bounded domain with* $\partial\Omega \in C^\infty$.
Let $b = {}^t(b_1, b_2, \cdots, b_n) \in C^1(\bar{\Omega})^n$ *be an irrotational vector, i.e.,*

$$\partial b_j/\partial x_i - \partial b_i/\partial x_j = 0 \quad \textit{for all } 1 \leqq i < j \leqq n.$$

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^n$;

(ii) $\{\mathrm{div}\ u_m + b \cdot u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)$, *and*
$\left\{\frac{\partial v_{m,i}}{\partial x_j} - \frac{\partial v_{m,j}}{\partial x_i} + v_{m,i}b_j - v_{m,j}b_i\right\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)$ *for all* $1 \leqq i < j \leqq n$;

(iii) *Either*
$\{u_m \cdot \nu\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$, *or*
$\{v_{m,i}\nu_j - v_{m,j}\nu_i\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$ *for all* $1 \leqq i < j \leqq n$.

$\Longrightarrow$

$$\int_\Omega u_m(x) \cdot v_m(x)dx \to \int_\Omega u(x) \cdot v(x)dx \quad \textit{as } m \to \infty.$$

4.2. Global Div-Curl lemma on Riemaniann manifolds with boundary

$(\bar{\Omega}, g)$: compact n-D Riemannian manifold with $\partial\Omega \in C^\infty$.

$$\tau u = \nu\rfloor(\nu \wedge u), \quad \nu u = \nu\rfloor u \quad \text{for } u \in \bigwedge^l(T\bar{\Omega}),\ 1 \le l \le n,$$

$\nu\rfloor : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l-1}(T\bar{\Omega})$: the interior product defined by

$$(\nu\rfloor u)(X_1, \cdots, X_{l-1}) = u(X_1, \cdots, X_{l-1}, \nu) \quad \text{for } X_1, \cdots, X_{l-1} \in T\bar{\Omega}.$$

$\Longrightarrow$

$$u = \tau u + \nu \wedge (\nu u) \quad \text{for all } u \in \bigwedge^l(T\bar{\Omega}).$$

$d : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l+1}(T\bar{\Omega})$: the exterior derivative, $l = 0, 1, \cdots, n-1$,
$* : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{n-l}(T\bar{\Omega})$: the Hodge star operator, $l = 0, 1, \cdots, n$,
$\delta : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l-1}(T\bar{\Omega})$: the codifferential operator, $l = 1, \cdots, n$,

$$\delta = (-1)^{n+1} * d * \chi^n, \quad \chi u = (-1)^l u \quad \text{for } u \in \bigwedge^l(T\bar{\Omega})$$

$$(u, v) \equiv \int_\Omega u \wedge *v, \quad \text{for } u, v \in \bigwedge^l(T\bar{\Omega}).$$

$$H_d(\Omega)^{l-1} \equiv \{u \in L^2(\Omega)^{l-1}; du \in L^2(\Omega)^l\},$$
$$H_\delta(\Omega)^l \equiv \{v \in L^2(\Omega)^l; \delta v \in L^2(\Omega)^{l-1}\}.$$

$$\exists \tau : u \in H_d(\Omega)^{l-1}(\Omega) \to \tau u \in H^{-\frac{1}{2}}(\partial\Omega)^{l-1},$$
$$\exists \nu : v \in H_\delta(\Omega)^l \to \nu v \in H^{-\frac{1}{2}}(\partial\Omega)^{l-1},$$

such that the generalized Stokes integral formula hollds:

$$(du, v) - (u, \delta v) = \langle \tau u, \nu v \rangle_{\partial\Omega}, \quad l = 1, \cdots, n,$$

$\forall \{u, v\} \in H_d(\Omega)^{l-1} \times H^1(\Omega)^l$ or $\forall \{u, v\} \in H^1(\Omega)^{l-1} \times H^1_\delta(\Omega)^l$.

Corollary 4 $(\bar{\Omega}, g)$*: n-D compact Riemannian manifold with $\partial\Omega \in C^\infty$.*
Let $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ be sequences of $L^2(\Omega)^l$, $l = 1, \cdots, n-1$ satisfying

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^l$;

(ii) $\{du_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{l+1}$, *and*
$\{\delta v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{l-1}$;

(iii) *Either*
$\{\tau u_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^l$, *or*
$\{\nu v_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{l-1}$.

$\Longrightarrow$

$$(u_m, v_m) \to (u, v) \quad \text{as } m \to \infty.$$

Theorem. *Let two pairs* $\{A_\alpha(x,D), A'_\alpha(x,D), B_\alpha(x,\nu), B'_\alpha(x,\nu)\}_{\alpha=1,2}$ *satisfy the generalized Stokes integral formula. Let the Assumption hold. We assume the cancellation property*

(4) $$A_2 A'_1 = 0, \quad A_1 A'_2 = 0.$$

Suppose that $\{u_m\}_{m=1}^\infty$ *and* $\{v_m\}_{m=1}^\infty$ *satisfy*

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^l$;

(ii) $\{A_1 u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_1}$ *and* $\{A_2 v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_2}$;

(iii) *Either* $\{B_1 u_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_1}$ *or* $\{B_2 v_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_2}$.

$\Longrightarrow$

$$\int_\Omega u_m(x)\cdot v_m(x)dx \to \int_\Omega u(x)\cdot v(x)dx \quad \text{as } m\to\infty.$$

Recall

the generalized Stokes formula

$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle B(\cdot, \nu)u, \varphi\rangle_{\partial\Omega},$$

$\forall u \in L^2(\Omega)^l$ with $A(x, D)u \in L^2(\Omega)^d$ and $\forall \varphi \in H^1(\Omega)^d$

$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle u, B'(\cdot, \nu)\varphi\rangle_{\partial\Omega},$$

$\forall \varphi \in L^2(\Omega)^d$ with $A'(x, D)\varphi \in L^2(\Omega)^l$ and $\forall u \in H^1(\Omega)^l$.

Assumption.

$$\begin{aligned}
\|\nabla u\| &\leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)}), \\
\|\nabla u\| &\leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_2 u\|_{H^{\frac{1}{2}}(\partial\Omega)})
\end{aligned}$$

for $\forall u \in H^1(\Omega)^l$. $\|\cdot\|$: L^2-norm on Ω.

Outline of the proof of Theorem.

In the case $\{B_1 u_m\}_{m=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^{d_1}$.

$S : L^2(\Omega)^l \mapsto L^2(\Omega)^{d_1+d_2}$, $D(S) = \{u \in H^1(\Omega)^l ; B_1 u = 0 \quad \text{on } \partial\Omega\}$,

$$Su \equiv^t (A_1 u, A_2 u) \quad \text{for } u \in D(S),$$

$\Longrightarrow$

$$\begin{aligned} D(S^*) = \quad & \{^t(p,w) \in L^2(\Omega)^{d_1} \times L^2(\Omega)^{d_2}; \\ & A_1' p \in L^2(\Omega)^l, A_2' w \in L^2(\Omega)^l, B_2' w = 0 \quad \text{on } \partial\Omega\} \end{aligned}$$

$$S^*(^t(p,w)) = A_1' p + A_2' w \quad \text{for } ^t(p,w) \in D(S^*),$$

Lemma 1 *(1)* dim.$Ker(S) < \infty$ *in* $L^2(\Omega)^l$.
(2) $R(S)$ *is closed in* $L^2(\Omega)^{d_1+d_2}$.

$\Longrightarrow$(closed range theorem)

$$L^2(\Omega)^l = Ker(S) \oplus R(S^*) \quad \text{(orthogonal decomposition)}.$$

$P : L^2(\Omega)^l \to Ker(S)$, $Q : L^2(\Omega)^l \to R(S^*)$: othogonal projections,

$$u = Pu + Qu, \quad v = Pv + Qv,$$
$$u_m = Pu_m + Qu_m, \quad v_m = Pv_m + Qv_m, \quad m = 1, 2, \cdots,$$

$\Longrightarrow$

$$(u_m, v_m) = (Pu_m, Pv_m) + (Qu_m, Qv_m), \quad m = 1, 2, \cdots.$$

P: finite rank operator since $R(P) = Ker(S)$) with dim.$Ker(S) < \infty$.

$\Longrightarrow$

$$Pu_m \to Pu, \quad Pv_m \to Pv \quad \textit{strongly} \text{ in } L^2(\Omega)^{d_1} \text{ as } m \to \infty,$$

$\Longrightarrow$

$$(Pu_m, Pv_m) \to (Pu, Pv) \quad \text{as } m \to \infty.$$

Hence it remains to show that

$$(Qu_m, Qv_m) \to (Qu, Qv) \quad \text{as } m \to \infty.$$

Note that

$$Qu = A_1' p + A_2' w, \quad Qv = A_1' \tilde{p} + A_2' \tilde{w}$$
$$Qu_m = A_1' p_m + A_2' w_m, \quad Qv_m = A_1' \tilde{p}_m + A_2' \tilde{w}_m, \quad m = 1, 2, \cdots.$$

$\Longrightarrow$ (by *cancellation property* $A_2 A_1' = 0$, $A_1 A_2' = 0$.)

$$(Qu, Qv) = (A_1' p, A_1' \tilde{p}) + (A_2' w, A_2' \tilde{w}),$$
$$(Qu_m, Qv_m) = (A_1' p_m, A_1' \tilde{p}_m) + (A_2' w_m, A_2' \tilde{w}_m),$$

Note that

$$B_1 A_2' w = B_1 A_2' \tilde{w} = B_1 A_2' w_m = B_1 A_2' \tilde{w}_m = 0 \quad \text{on } \partial\Omega$$

Proposition. *(1)* $\{A_1' p_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.

(2) $\{A_2' \tilde{w}_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.

Proof. $A_2 A_1' = 0$ & $\sup_{m=1,2,\cdots} \|B_1 u_m\|_{H^{\frac{1}{2}}(\partial\Omega)} < \infty$

$\Longrightarrow$

$$\|\nabla(A_1' p_m)\| \leqq C(\|A_1 u_m\| + \|u_m\| + \|B_1 u_m\|_{H^{\frac{1}{2}}(\partial\Omega)}) \leqq \exists M, \quad \forall m = 1, 2, \cdots.$$

$A_1 A_2' = 0$

$\Longrightarrow$

$$\|\nabla(A_2' \tilde{w}_m)\| \leqq C(\|A_2 v_m\| + \|v_m\|) \leqq \exists M, \quad \forall m = 1, 2, \cdots.$$

1. Gérard, P., *Miclolacal defect measures.* Commun. Partial Differential Equations **16**, 1761–1794 (1991).

2. Kazhikhov,A.V., *Approximation of weak limits and related problems.* "Mathematical Foundation of Turbulent Viscous Flows". Cannone, M., Miyakawa, T., ed. Springer-Verlag LNM CIME **1871**, 75–100 (2006).

3. Kozono,H., Yanagisawa,T., *Global DIV-CURL Lemma in bounded domains in* $\mathbb{R}^3$*.* Jour. Funct. Anal. **256**, 3847–3859 (2009).

4. Murat, F., *Compacité par compensation II.* Proceeding of the International Meeting on Recent Methods in Nonlinear Analysis, ed. by E. de Giorgi, E. Magenes, U. Mosco, 245–256, Pitagora, Bologna, 1979

5. Murat, F., *Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothése de rang constant.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8**, 69–102 (1981).

6. Tartar,L., *Compensated compactness and applications to partial differential equations, Nonlinear Analysis and Mechanics*: Heriot-Watt Symposium **4**, R.J. Knops, ed., Research Notes in Math., Pitman, 1979.

7. Tartar,L., H*-measures, a new approach for studying homogenisation, osclla-tion and concentration effect in partial differential equations.* Proc. Roy. Soc. Edinburgh Sect. A **115**, 193–230 (1990).

8. Tartar,L., *Compensation effects in partial differential equations.* Memorie di Matematica e Applicazioni, Rendiconti della Accademia Nazionale delle Scienze detta dei XL, Ser. V, vil XXIX, 395–454 (2005).

Thank you for your attention!