Optimal initial value conditions for local strong solutions of the Navier-Stokes equations in exterior domains (and applications to regularity theory)

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Problem statement I

Let $\Omega \subseteq \mathbb{R}^3$ be a domain and $[0, T[$ be a time interval. We consider on $[0, T[\times \Omega$ the instationary Navier-Stokes equations

$$u_t - \nu \Delta u + u \cdot \nabla u + \nabla p = f\,, \qquad \text{div } u = 0\,,$$

$$u(0) = u_0\,, \qquad u|_{\partial\Omega} = 0.$$

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The existence of global weak solutions

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$$

on $[0,T[\times\Omega$ for an initial value $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \mathrm{div} F$ with $F \in L^2(0,T;L^2(\Omega))$ is well known.

Problem statement II

A central question is the existence of a strong strong u. This means a weak solution u which fulfills the Serrin condition

$$u \in L^s(0,T;L^q(\Omega))\ , \quad 2 < s < \infty\, , \frac{2}{s} + \frac{3}{q} = 1.$$

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The existence of global strong solutions is closely related to one of the seven Millenium Problems of Clay Mathematics Institute.

Problem statement III

Up to now, the existence of a strong solution u could only be proven in a sufficiently small interval $[0, T[\,, 0 < T < \infty$, and under additional assumptions on $\Omega\,, f$ and u_0.
In the case that Ω is a smooth bounded domain it was proved by Farwig, Sohr and Varnhorn that the condition

$$\int_0^\infty \|e^{-\nu\tau A} u_0\|_q^s \, d\tau < \infty$$

is necessary and sufficient for the existence of a unique strong solution of the N-S-e in some interval $[0, T[\,, 0 < T < \infty$. As usual $A = A_2$ denotes the Stokes operator on $L^2_\sigma(\Omega)$.

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is necessary and sufficient for the existence of a unique strong solution of the N-S-e in some interval $[0,T[\,, 0<T<\infty$. As usual $A=A_2$ denotes the Stokes operator on $L^2_\sigma(\Omega)$. We want to transform this result to the case that Ω is an exterior domain. Further we will apply this result to obtain regularity criteria for weak solutions of the N-S-e.

General Notation

For exponents $1 < s, q < \infty$ we define the Serrin number by

$$\mathcal{S}(s,q) := \frac{2}{s} + \frac{3}{q}.$$

Throughout this talk, let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$. Unless explicitly stated let $0 < T < \infty\,, \nu > 0$ and let $1 < s\,, s_*\,, q\,, q_* < \infty$ with

$$\mathcal{S}(s,q) = 1\,, \mathcal{S}(s_*, q_*) = 2$$

where

$$\frac{1}{q} \leq \frac{1}{q_*} \leq \frac{1}{3} + \frac{1}{q}.$$

Sufficient criterion for local strong solutions

Theorem

Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < s, s_*, q, q_* < \infty$ with $\mathcal{S}(s,q) = 1\,, \mathcal{S}(s_*, q_*) = 2$ where $\frac{1}{q} \leq \frac{1}{q_*} \leq \frac{1}{3} + \frac{1}{q}$. Let $0 < T < \infty\,, \nu > 0$, let $F \in L^{s_*}(0,T; L^{q_*}(\Omega)) \cap L^2(0,T; L^2(\Omega))$, and $u_0 \in L^2_\sigma(\Omega)$. If $q \in [\frac{24}{7}, 8]$ then there exists a constant $\epsilon_* = \epsilon_*(\Omega, q, q_*) > 0$ such that the conditions

$$\int_0^T \|e^{-\nu\tau A} u_0\|_q^s \, d\tau \leq \epsilon_* \nu^{s-1}\,,$$

$$\int_0^T \|F(\tau)\|_{q_*}^{s_*} \, d\tau \leq \epsilon_* \nu^{s_*(1+\frac{3}{2q_*})}\,,$$

are sufficient for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ of the N-S-e satisfying the energy equality.

Sketch of proof I

Let $u = E + \tilde{u}$, where

$$E(t) = e^{-\nu t A} u_0 + A_{q*}^{1/2} \int_0^t e^{-\nu(t-\tau)A_{q*}} (A_{q*}^{-1/2} P_q \,\text{div}\, F(\tau))\, d\tau$$

for almost all $t \in [0, T[$ and $\tilde{u} \in L^s(0, T; L^q_\sigma(\Omega))$ satisfies almost everywhere the fixed point equation

$$\tilde{u}(t) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \,\text{div}\, ((\tilde{u} + E) \otimes (\tilde{u} + E))\, d\tau.$$

with $\alpha := \frac{1}{2} + \frac{3}{2q}$. The motivation is that $u \in L^s(0, T; L^q_\sigma(\Omega))$ is a very weak solution .

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$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_{0,\sigma}(\Omega)).$$

This implies that u is a weak solution.

Sketch of proof II

From the linear theory of the N-S-e we get that E is also a weak solution. The major point in the proof of this theorem is to show $\tilde{u} \in L^8(0, T; L^4(\Omega))$. After proving this statement we can conclude

$$u \otimes u = (\tilde{u} + E) \otimes (\tilde{u} + E) \in L^2(0, T; L^2(\Omega)).$$

Then $\tilde{u}$ can be considered as solution of the linear instationary Stokes system with initial value 0 and external force with divergence part $u \otimes u \in L^2(0, T; L^2(\Omega))$. Therefore, by linear theory, u is also a weak solution.

Optimal initial value conditions

Theorem

Let $\Omega \subseteq \mathbb{R}^3$ *be an exterior domain with* $\partial\Omega \in C^{2,1}$, *let* $1 < s, s_*, q, q_* < \infty$ *with* $\mathcal{S}(s,q) = 1, \mathcal{S}(s_*, q_*) = 2$ *where* $\frac{1}{q} \leq \frac{1}{q_*} \leq \frac{1}{3} + \frac{1}{q}$. *Let* $0 < T < \infty, \nu > 0$, *let* $F \in L^{s_*}(0,T; L^{q_*}(\Omega)) \cap L^2(0,T; L^2(\Omega))$, *and* $u_0 \in L^2_\sigma(\Omega)$. *If* $q \in [\frac{24}{7}, 8]$ *then the condition*

$$\int_0^\infty \|e^{-\nu\tau A} u_0\|_q^s \, d\tau < \infty$$

is necessary and sufficient for the existence of a strong solution of the N-S-e in an interval $[0, T'[$ *with* $0 < T' \leq T$.

Introduction to regularity criteria I

Let

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$$

be a weak solution of the instationary N-S-e. We say that u is (locally) regular at $t \in]0,T[$ if there is $\delta = \delta(t) > 0$ such that $u \in L^s(t-\delta, t+\delta; L^q(\Omega))$ with any Serrin exponents $\mathcal{S}(s,q) = 1$. In the rest of this talk we want to formulate regularity criteria for u. This means conditions on the data f , u_0 and the solution u itself such that u is locally or globally regular.

Introduction to regularity criteria II

In the following we consider a weak solution u of the instationary N-S-e satisfying the strong energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \leq \frac{1}{2}\|u(s)\|_2^2 - \int_s^t \langle F(\tau), \nabla u(\tau)\rangle_\Omega$$

for almost all $s \in [0, T[$ and all $t \in [s, T[$.
Fix $1, s_*, q, q_* < \infty$ such that

$$\mathcal{S}(s,q) = 1\,, \quad \mathcal{S}(s_*, q_*) = 2$$

where

$$\frac{1}{q} \leq \frac{1}{q_*} \leq \frac{1}{3} + \frac{1}{q}.$$

Further fix an exterior domain $\Omega \subseteq \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$ and $0 < T < \infty\,, \nu > 0$.

Local in time regularity

Theorem

Let $1 \leq s' \leq s\,, u_0 \in L^2_\sigma(\Omega)\,, f = \mathrm{div} F$ *with* $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, u *a weak solution of the N-S-e, satisfying the strong energy inequality.*

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- *If* $u \in L^s(t-\delta, t; L^q(\Omega))$ *with* $\delta > 0$, *then* u *is regular at* t.

Local in time regularity

Theorem

Let $1 \le s' \le s\,, u_0 \in L^2_\sigma(\Omega)\,, f = \mathrm{div} F$ *with* $F \in L^{s_*}(0,T; L^{q_*}(\Omega)) \cap L^2(0,T; L^2(\Omega))$, u *a weak solution of the N-S-e, satisfying the strong energy inequality.*

- *If* $u \in L^s(t-\delta, t; L^q(\Omega))$ *with* $\delta > 0$, *then* u *is regular at* t.
- *The following condition is sufficient and necessary for regularity of* u *at* t:

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^{t} \|u(\tau)\|_q^{s'}\, d\tau = 0.$$

Global in time regularity

Theorem

Let $1 \le s' \le s$, $u_0 \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$, $f = \mathrm{div} F$ *with* $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, u *a weak solution of the N-S-e satisfying the strong energy inequality.*

- $\exists \epsilon_* = \epsilon_*(q, q_*, \Omega, s') > 0$, *such that the conditions*

$$\int_0^T \|F(\tau)\|_{q_*}^{s_*}\, d\tau \le \epsilon_* \nu^{s_*(1+\frac{3}{2q_*})} ,$$

$$\int_0^T \|u(\tau)\|_q^{s'}\, d\tau \le \epsilon_* \frac{\nu^{s-1}}{\|u_0\|_q^{s-s'}}$$

imply $u \in L^s(0,T;L^q(\Omega))$.

Energy-based criteria, case $\alpha > \frac{1}{2}$

Theorem

Let $\frac{2}{s'} + \frac{3}{q} = \frac{3}{2}$, $u_0 \in L^2_\sigma(\Omega)$, $f \in L^{\frac{s'}{s}}(0,T;L^q(\Omega))$ *with* $f = \mathrm{div} F$, $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, u *a weak solution of the N-S-e satisfying the strong energy inequality with in time continuous kinetic energy* $E(t) = \frac{1}{2}\|u(t)\|_2^2$.

Energy-based criteria, case $\alpha > \frac{1}{2}$

Theorem

Let $\frac{2}{s'} + \frac{3}{q} = \frac{3}{2}\,, u_0 \in L^2_\sigma(\Omega)\,, f \in L^{\frac{s'}{s}}(0,T;L^q(\Omega))$ *with* $f = \operatorname{div} F\,, F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))\,, u$ *a weak solution of the N-S-e satisfying the strong energy inequality with in time continuous kinetic energy* $E(t) = \frac{1}{2}\|u(t)\|_2^2$.
For $\alpha \in]\frac{1}{2}, 1[$ *each of the conditions*

$$\sup_{t-\mu<t'<t} \frac{1}{\delta^\alpha}|E(t) - E(t')| < \infty\,, \quad \textit{with a } \mu > 0\,,$$

$$\liminf_{\delta\to 0+} \frac{1}{\delta^\alpha} \int_{t-\delta}^{t} \|\nabla u(\tau)\|_2^2\, d\tau < \infty$$

imply regularity of u *at* t.

Energy-based criteria, case $\alpha = \frac{1}{2}$

Theorem

Let $u_0 \in L^2_\sigma(\Omega)\,,\, f \in L^2(0,T;L^2(\Omega))\,,\, f = \mathrm{div} F$ *with* $F \in L^4(0,T;L^2(\Omega))\,,\, u$ *a weak solution of the N-S-e satisfying the strong energy inequality with in time continuous kinetic energy* $E(t) = \frac{1}{2}\|u(t)\|_2^2$.
There exists $\gamma_* = \gamma_*(\Omega)$ *such that each of the conditions*

$$\sup_{t-\mu<t'<t} \frac{1}{\delta^{\frac{1}{2}}}|E(t) - E(t')| \leq \gamma_* \nu^{\frac{5}{2}}\,, \quad \text{with } \mu > 0\,,$$

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{\frac{1}{2}}} \int_{t-\delta}^{t} \|\nabla u(\tau)\|_2^2 \, d\tau \leq \gamma_* \nu^{\frac{3}{2}}$$

imply regularity of u *at* t.

Sketch of proof of the regularity criteria

Main idea. We consider $u(t)$ for almost all $t \in [0, T[$ as the initial value of a strong solution v and identify u and v.
Assume we want to show $u \in L^s(a, b; L^q(\Omega))$. We construct a strong solution $v \in L^s(0, T; L^q_\sigma(\Omega))$ with $v(a) = u(a)$. Since u satisfies the strong energy inequality Serrin's uniqueness theorem yields that u and v must coincide and therefore $u \in L^s(a, b; L^q(\Omega))$.

Thank you!