Function spaces with variable smoothness and integrability

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September 19, 2011

Overview

Main Result

Variable exponent function spaces

Function spaces with also $q(\cdot)$ variable

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Mathematical Folklore

Theorems are easier to prove in Besov $B^s_{p,q}(\mathbb{R}^n)$ spaces than in Triebel-Lizorkin $F^s_{p,q}(\mathbb{R}^n)$ spaces.
(e.g. Characterizations by local means, atoms ....)

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This statement reverses if the indices $s(\cdot), p(\cdot), q(\cdot)$ are variable functions of the space variable $x \in \mathbb{R}^n$.

Variable exponent function spaces

Variable exponent Lebesgue spaces

Let $p : \mathbb{R}^n \to [1, \infty]$ measurable. Then

$$\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}, \mathbb{R}^n_0 = \mathbb{R}^n \setminus \mathbb{R}^n_\infty$$

and

$$p^- = \operatorname*{ess-inf}_{x \in \mathbb{R}^n} p(x) \text{ and } p^+ = \operatorname*{ess-sup}_{x \in \mathbb{R}^n} p(x) .$$

For $f : \mathbb{R}^n \to \mathbb{C}$ measurable, we define the **convex modular**

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n_0} |f(x)|^{p(x)} dx + \operatorname*{ess-sup}_{x \in \mathbb{R}^n_\infty} |f(x)| .$$

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$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n_0} |f(x)|^{p(x)} dx + \operatorname*{ess-sup}_{x\in\mathbb{R}^n_\infty} |f(x)| \, .$$

Then $L_{p(\cdot)}(\mathbb{R}^n)$ is the collection of all f such that there exists an $\lambda > 0$ with $\varrho_{p(\cdot)}(f/\lambda) < \infty$.
$L_{p(\cdot)}(\mathbb{R}^n)$ is a Banachspace with the norm
$\big\| f | \, L_{p(\cdot)}(\mathbb{R}^n) \big\| = \inf\{\lambda > 0 : \varrho_p(f/\lambda) \le 1\}$.

$\log$-*Hölder regularity*

Definition. *Let* $g \in C(\mathbb{R}^n)$*. We say that* g *is* locally $\log$-Hölder continuous, *abbreviated* $g \in C^{\log}_{loc}(\mathbb{R}^n)$*, if there exists* $c_{\log}(g) > 0$ *such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)}$$

holds for all $x, y \in \mathbb{R}^n$*.*

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$$|g(x) - g_\infty| \leq \frac{c}{\log(e + |x|)}$$

holds for all $x \in \mathbb{R}^n$.

Admissible weight sequence

Definition. *Let $\alpha \geq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$. We say that a sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if and only if*

1. *There exists a constant $C > 0$ such that*

$$0 < w_j(x) \leq C w_j(y) \left(1 + 2^j |x-y|\right)^{\alpha} \quad j \in \mathbb{N}_0 \text{ and } x, y \in \mathbb{R}^n.$$

2. *For all $j \in \mathbb{N}_0$ we have*

$$2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ is called admissible weight sequence.

Resolution of Unity

Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ with $\varphi_0(x) \geq 0$ and

$$\varphi_0(x) = \begin{cases} 1 \text{ , if } |x| \leq 1 \\ 0 \text{ , if } |x| \geq 2 \end{cases} .$$

We define $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) = \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$, then we obtain

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all } x \in \mathbb{R}.$$

Definition of the spaces

Definition. *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a resolution of unity. Furthermore let* $0 < q \leq \infty$ *and* $1/p \in C^{\log}(\mathbb{R}^n)$*.*

- *Then for* $0 < p^- \leq p^+ \leq \infty$

$$B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_{\varphi} < \infty \right\} \quad \textit{with}$$

$$\|f|B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_{\varphi} = \left(\sum_{j=0}^{\infty} \|(\varphi_j \hat{f})^{\vee} w_j | L_{p(\cdot)}(\mathbb{R}^n)\|^q \right)^{1/q} .$$

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Properties

- Independence of the chosen resolution of unity $\{\varphi_j\}$
- Characterizations by:
  - local means
  - atoms, molekules and wavelets
  - differences [Besov03] ($1 \leq p, q \leq \infty$ and $\alpha_1 \geq 0$)
- Invariance under diffeomorphisms
- Pointwise multiplier assertion

Remark: This properties can be proved by the standard methods, since for $1/p(\cdot) \in C^{\log}(\mathbb{R}^n)$ the maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ and on $L_{p(\cdot)}(\ell_q)$.

Connection to known spaces

Let $p(\cdot) = p \in (0, \infty]$ ($p < \infty$ in F-case)

- $w_j(x) = 2^{js} \rightsquigarrow$ classical Besov and Triebel-Lizorkin spaces with Sobolev and Hölder-Zygmund spaces included (Triebel,...)
- $w_j(x) = 2^{js}(1 + 2^j|x - x_0|)^{s'} \rightsquigarrow$ 2-microlocal spaces $C^{s,s'}_{x_0} = B^{\boldsymbol{w}}_{\infty,\infty}(\mathbb{R}^n)$ and $H^{s,s'}_{x_0} = B^{\boldsymbol{w}}_{2,2}(\mathbb{R}^n)$ (Jaffard, Meyer,...)
- $w_j(x) = 2^{\sigma_j}$ with $d_1\sigma_j \le \sigma_{j+1} \le d_2\sigma_j \rightsquigarrow$ Function spaces of generalized smoothness $B^{\{\sigma_j\}}_{p,q}(\mathbb{R}^n)$, $F^{\{\sigma_j\}}_{p,q}(\mathbb{R}^n)$ (Goldman, Kalyabin, Leopold, Moura, ...)
- $w_j(x) = 2^{js(x)}$ with $0 < s^- \le s^+ \le 1 \rightsquigarrow$ variable Hölder-Zygmund spaces $\mathcal{C}^{s(\cdot)} = B^{\boldsymbol{w}}_{\infty,\infty}(\mathbb{R}^n)$ (Almeida, Samko)

Let $1 < p^- \leq p^+ < \infty$

- $w_j(x) = 2^{js}$ with $s \in \mathbb{R} \rightsquigarrow$ variable Bessel potential spaces $\mathcal{L}^s_{p(\cdot)} = F^w_{p(\cdot),2}(\mathbb{R}^n)$ (Almeida, Samko, Nekvinda, Harjulehto, Gurka, ...)
- $w_j(x) = 2^{jk}$ with $k \in \mathbb{N}_0 \rightsquigarrow$ variable Sobolev spaces $W^k_{p(\cdot)} = F^w_{p(\cdot),2}(\mathbb{R}^n)$ (Kovacik, Rakosnik, ...)

Connection to spaces of variable smoothness

Lemma. *Let* $s : \mathbb{R}^n \to \mathbb{R}$ *be a measurable function. Then* $s \in C^{\log}_{loc}(\mathbb{R}^n)$ *if, and only if,* $\{w_j\}_{j\in\mathbb{N}_0}$ *defined by*

$$w_j(x) = 2^{js(x)} \tag{1}$$

belongs to $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *with* $\alpha_1 = s^-$, $\alpha_2 = s^+$ *and* $\alpha = c_{\log}(s)$.

That means with $w_j(x) = 2^{js(x)}$ we have

$$B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n) \quad \text{and} \quad F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = F^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n).$$

But: Not every admissible weight sequence $\{w_j\} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ can be expressed by a smoothness function $s(\cdot)$ via (1).
Example: $w_j(x) = 2^{js}(1 + 2^j|x - x_0|)^{s'}$ with $s' \neq 0$.

Function spaces with also $q(\cdot)$ variable

Motivation

By tr we denote the trace operator from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$.

Theorem (Diening, Hästö,Roudenko). *Let* $s(\cdot) \in C^{\log}_{loc}$ *and* $1/p(\cdot), 1/q \in C^{\log}$ *with* $0 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ < \infty$. *If*

$$s(x) - \frac{1}{p(x)} - (n-1)\left(\frac{1}{p(x)} - 1\right)^+ > 0\,,$$

$$\textit{then} \quad tr F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1})\,.$$

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Observation: If $p(\cdot)$ is variable, then s and q must be variable too to obtain trace results in one scale.

Definition of F-spaces

Definition. *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a resolution of unity. Furthermore let* $\frac{1}{p(\cdot)}, \frac{1}{q(\cdot)} \in C^{\log}(\mathbb{R}^n)$ *be variable exponents with* $0 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ < \infty$. *We define*

$$F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_{\varphi} < \infty \right\} \quad \textit{with}$$

$$\|f|F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_{\varphi} = \left\| \left(\sum_{j=0}^{\infty} |(\varphi_j \hat{f})^{\vee}(\cdot) w_j(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \Bigg| L_{p(\cdot)}(\mathbb{R}^n) \right\| .$$

Remark: These spaces are independent of the resolution of unity $\{\varphi_j\}$ and are Banach spaces for $\min(p(x), q(x)) \geq 1$ pointwise.

Another modular

We have to define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ by another modular:

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \leq 1\right\}. \quad (2)$$

If $q^+ < \infty$, then we can replace (2) by the simpler expression

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_\nu \left\| \left|f_\nu\right|^{q(\cdot)} \Big| L_{\frac{p(\cdot)}{q(\cdot)}} \right\|.$$

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The (quasi-)norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usual by

$$\| f_\nu | \ell_{q(\cdot)}(L_{p(\cdot)})\| = \inf\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \leq 1\}.$$

The norm question

$\ell_{q(\cdot)}(L_{p(\cdot)})$ is for all $p(\cdot), q(\cdot)$ a quasi-normed space.

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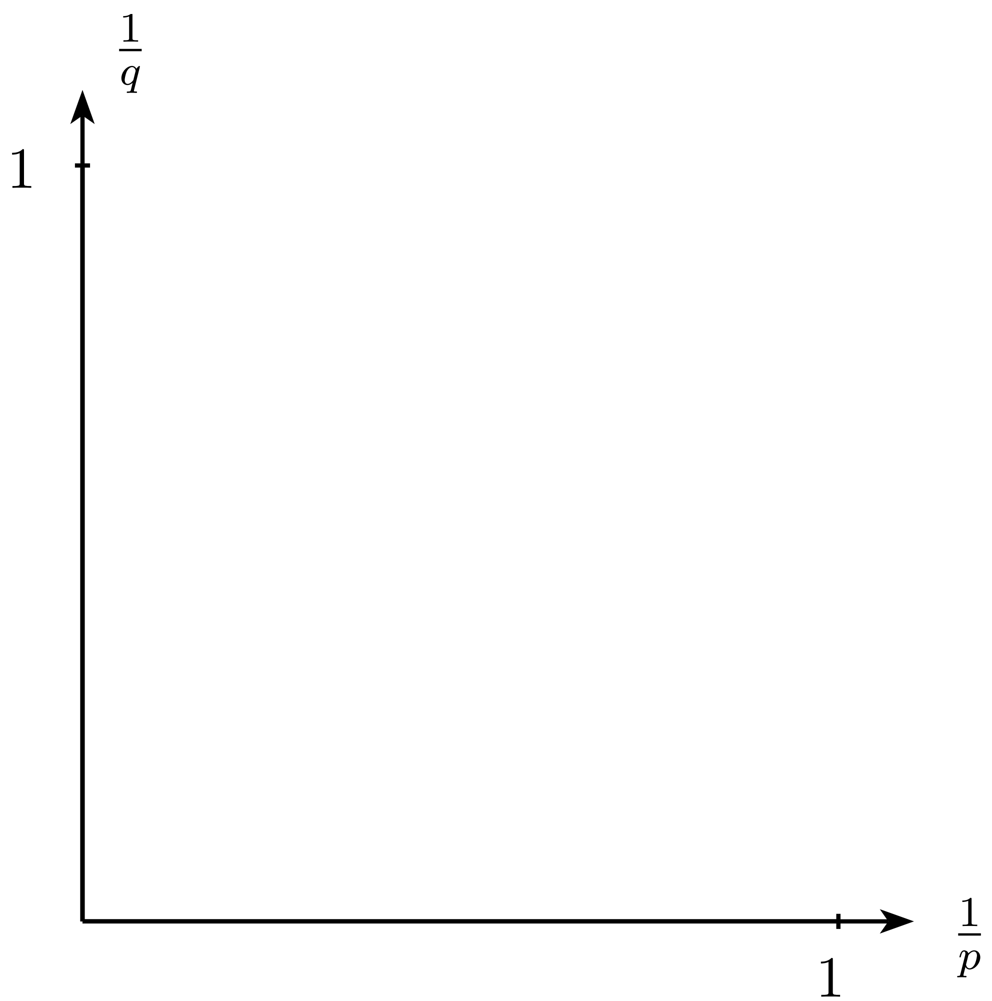
- Counterexample: There exist functions $p(x) \geq 1$ and $q(x) \geq 1$ such that $\ell_{q(\cdot)}(L_{p(\cdot)})$ is not a normed space. (K., Vybíral)

Counterexample

- Q_0, Q_1 are two disjoint unit cubes
- $p(x) = 1$ everywhere
- $q(x) = \infty$ for $x \in Q_1$ and $q(x) = 1$ for $x \in Q_0$
- Set $f_0 = \chi_{Q_0}$ and $f_1 = \chi_{Q_1}$ and $f = (f_0, f_1, 0, \dots)$, $g = (f_1, f_0, 0, \dots)$
- Then $\big\| f |\, \ell_{q(\cdot)}(L_{p(\cdot)}) \big\| = \big\| g |\, \ell_{q(\cdot)}(L_{p(\cdot)}) \big\| = 1$ but $\big\| f + g |\, \ell_{q(\cdot)}(L_{p(\cdot)}) \big\| = 3$

Norm Question?



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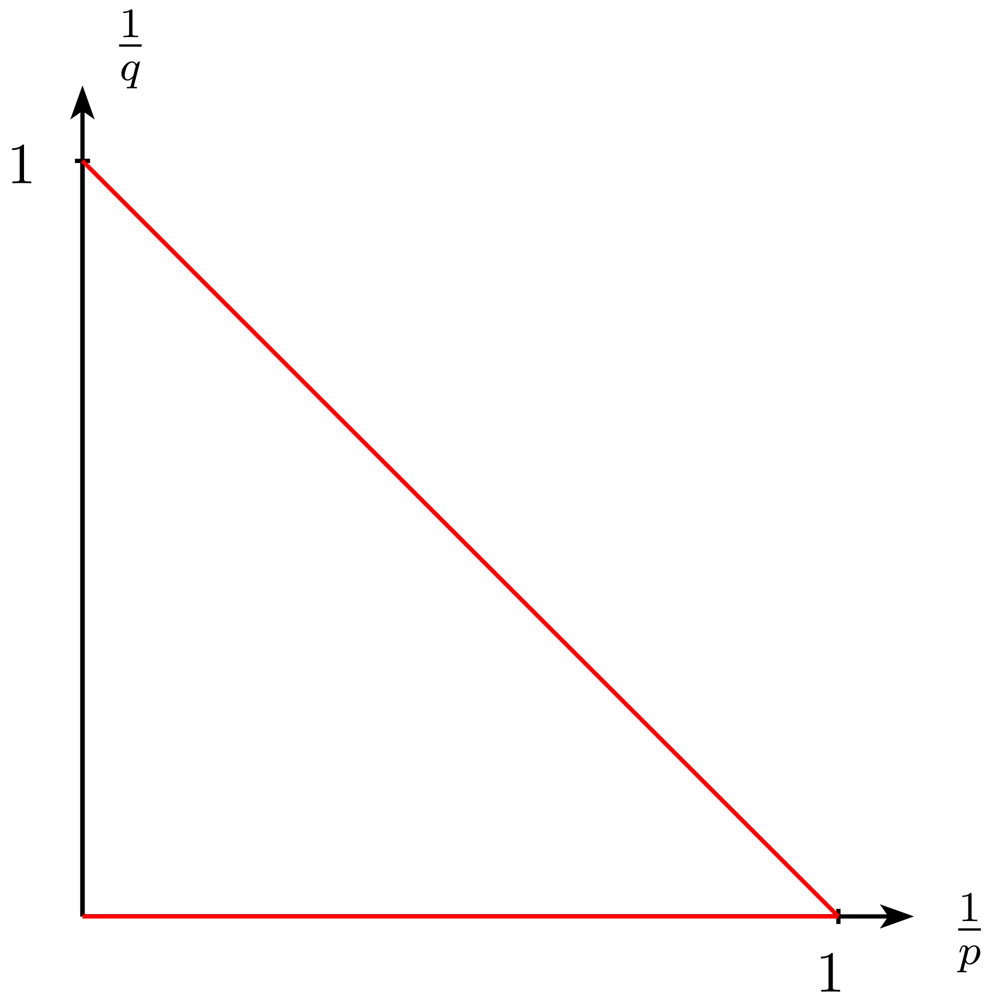

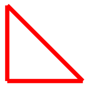

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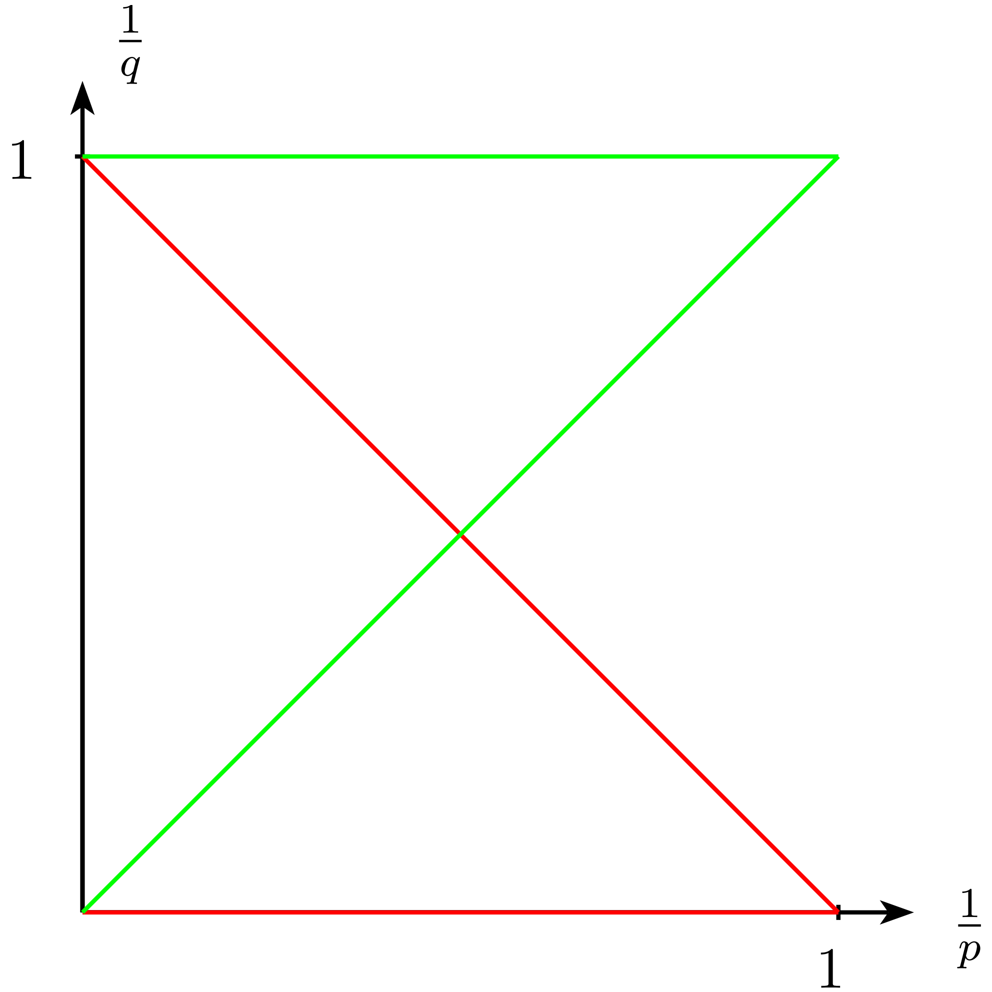

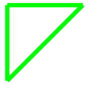

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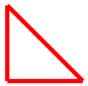

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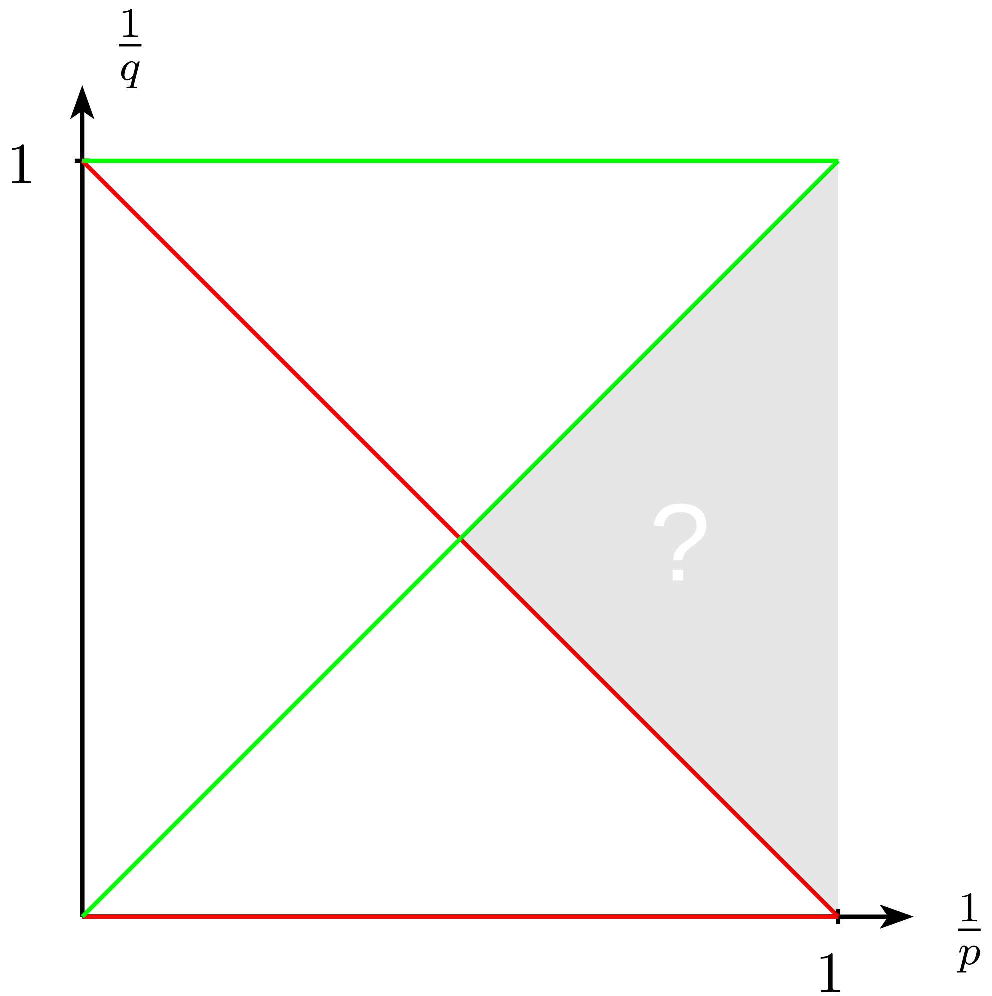

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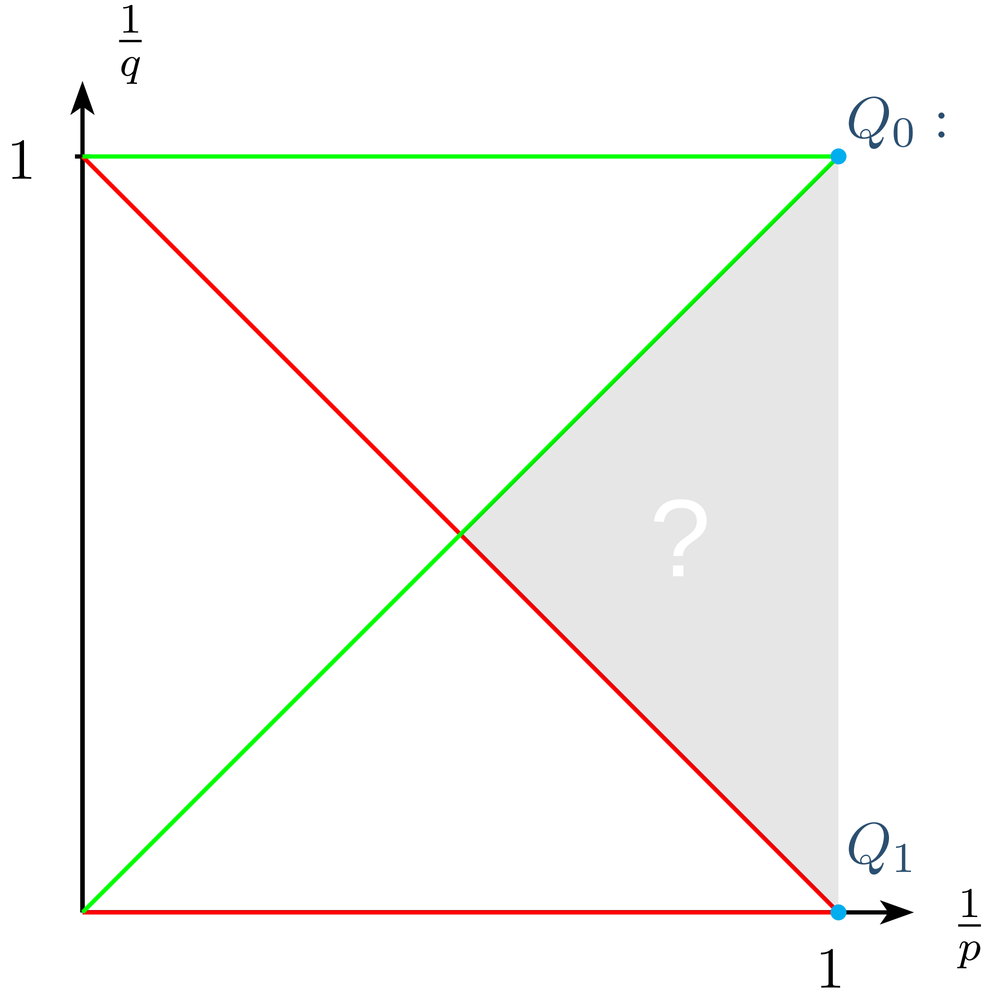
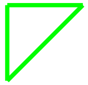


Q_1 : $p = 1, q = \infty$
Q_0 $p = 1, q = 1$



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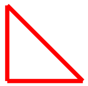

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$$B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f |\, B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_{\varphi} < \infty \right\},$$

where

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Remark: These spaces are independent of the chosen resolution of unity.

By setting $w_j(x) = 2^{js(x)}$ with $s \in C^{\log}_{loc}$ we obtain the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and also $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Maximal operator

The maximal operator $\mathcal{M}$ is not bounded on $\ell_{q(\cdot)}(L_{p(\cdot)})$ and $L_{p(\cdot)}(\ell_{q(\cdot)})$ if $q(\cdot)$ is a variable function.

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The maximal operator $\mathcal{M}$ is not bounded on $\ell_{q(\cdot)}(L_{p(\cdot)})$ and $L_{p(\cdot)}(\ell_{q(\cdot)})$ if $q(\cdot)$ is a variable function.
Replacement: Consider the functions
$\eta_{\nu,m}(x) = 2^{\nu n}(1+2^{\nu}|x|)^{-m}$, then for $m > n$

$$\left\| (\eta_{\nu,m} * f_\nu)_{\nu\in\mathbb{N}_0} \middle| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \leq c \left\| f_\nu \middle| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

and for $m > n + c_{\log}(1/q)$

$$\left\| (\eta_{\nu,m} * f_\nu)_{\nu\in\mathbb{N}_0} \middle| \ell_{q(\cdot)}(L_{p(\cdot)}) \right\| \leq c \left\| f_\nu \middle| \ell_{q(\cdot)}(L_{p(\cdot)}) \right\|$$

With that convolution inequalities we proved a characterization by differences for $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ (resp. $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$) with $0 < p^- \leq p^+ \leq \infty$ and $0 < q^- \leq q^+ \leq \infty$
$\rightsquigarrow$ talk of Jan Vybíral.

Properties (Almeida, Hästö)

- If $q(\cdot)$ is constant, then $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.
- If $p(\cdot) = q(\cdot)$, then $F^{\boldsymbol{w}}_{p(\cdot),p(\cdot)}(\mathbb{R}^n) = B^{\boldsymbol{w}}_{p(\cdot),p(\cdot)}(\mathbb{R}^n)$.
- If $p^+, q^+ < \infty$, then

$$B^{\boldsymbol{w}}_{p(\cdot),\min(p(\cdot),q(\cdot))} \hookrightarrow F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{\boldsymbol{w}}_{p(\cdot),\max(p(\cdot),q(\cdot))} .$$

- If $(s_0 - s_1)^- > 0$, then $B^{s_0(\cdot)}_{p(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s_1(\cdot)}_{p(\cdot),q_1(\cdot)}(\mathbb{R}^n)$
- If $s_0(\cdot) \geq s_1(\cdot)$, $q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)} \in C^{\log}_{loc} \text{ , then}$$

$$B^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

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